NASA Contractor Report 187563

ICASE Report No. 91–39

ICASE

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INSTABILITY OF GÖRTLER VORTICES IN
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NONLINEAR DEVELOPMENT AND SECONDARY INSTABILITY OF GÖRTLER VORTICES IN HYPERSONIC FLOWS [1]

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Abstract

In a hypersonic boundary layer over a wall of variable curvature, the region most susceptible to Görtler vortices is the temperature adjustment layer over which the basic state temperature decreases monotonically to its free stream value (Hall & Fu (1989), Fu, Hall & Blackaby (1990)). Except for a special wall curvature distribution, the evolution of Görtler vortices trapped in the temperature adjustment layer will in general be strongly affected by boundary layer growth through the $O(M^{3/2})$ curvature of the basic state, where M is the free stream Mach number. Only when the local wavenumber becomes as large as of order $M^{3/8}$, do nonparallel effects become negligible in the determination of stability properties. In the latter case, Görtler vortices will be trapped in a thin layer of $O(\epsilon^{1/2})$ thickness which is embedded in the temperature adjustment layer; here ϵ is the inverse of the local wavenumber. In this paper, we first present a weakly nonlinear theory in which the initial nonlinear development of Görtler vortices in the neighbourhood of the neutral position is investigated and two coupled evolution equations are derived. From these two evolution equations we can determine whether the vortices are decaying or growing depending on the sign of a constant which is related to the wall curvature and the basic state temperature. In the latter case, it is found that the mean flow correction becomes as large as the basic state at distances $O(1)$ downstream of the neutral position. Next, we present a fully nonlinear theory concerning the further downstream development of these large-amplitude Görtler vortices. It is shown that the vortices spread out across the boundary layer. The upper and lower boundaries of the region of vortex activity are determined by a free-boundary problem involving the boundary layer equations. Finally, the secondary instability of the flow in the transition layers located at the upper and lower edges of the the region of vortex activity is considered. The superimposed wavy vortex perturbations are spanwise periodic travelling waves which are $\pi/2$ radians out of phase with the fundamental. The dispersion relation is found to be determined by solving two coupled differential equations and it is shown that an infinite number of neutrally stable modes may exist.

1 Introduction

This paper is the third of a series of papers reporting on our studies of the stability properties of hypersonic boundary layers with respect to the Görtler instability mechanism. The previous two papers have been devoted to the linear development of Görtler vortices in hypersonic boundary layers. The first, Hall and Fu (1989), is concerned with fluids which have their viscosity modelled by Chapman's law; whilst the second, Fu, Hall and Blackaby (1990), studies Sutherland's law fluids. In the second paper, the roles played by gas dissociation and wall cooling in the determination of stability properties are also clarified. In the present paper, we first study the nonlinear development of Görtler vortices in the neighbourhood of the neutral position and show how a large amplitude vortex structure can be developed under the combined effects of viscosity and nonlinearity. Then we consider one of the several possible types of secondary instabilities which the latter vortex structure may suffer, namely, the wavy type. For a review of the general literature on Görtler instability in hypersonic flows we refer the reader to our previous papers. A detailed review for the related incompressible flow problems can be found in Hall (1988). Here we only mention the papers which are most relevant to our present studies.

The type of nonlinear theory which we use here is different from the classical weakly nonlinear theory and was first established by Hall (1982) and Hall and Lakin (1988) in the context of Görtler instability in incompressible flows. In the classical weakly nonlinear theory, see, for examle, Stuart (1965), the size of the disturbance is chosen such that the cumulative effects of nonlinearity are brought into the evolution equation as a solvability condition at the third order of a successive approximation procedure using a multiple scales approach. This is possible because the growth rate in the neighbourhood of the neutral position is small. Hall's (1982b) nonlinear theory is suitable for problems which have large growth rate (corresponding to large wavenumbers); the size of the disturbance is chosen to be so large that nonlinear effects come into play at the order at which the vertical structure is determined. Such large amplitude Görtler vortices can grow downstream of the neutral position and become so large as to produce a mean flow correction as large as the basic state, as is shown by Hall and Lakin (1985). When this happens, the vortices are confined to a core region bounded by two transition layers. In the core region of Görtler vortex activity, the boundary layer is forced by the vortex which itself is driven by the boundary layer. In the two transition layers, the vortex is reduced to zero exponentially. Hall and Seddougui (1989) and Seddougui and Bassom (1990) studied the wavy type of secondary instability which might occur in the two transition layers. Their studies were motivated by the experimental results of Bippes (1978) and Aihara and Koyama (1981), who observed that the three-dimensional breakdown of steady spanwise periodic Görtler vortices led to a time periodic flow with wavy vortex boundaries similar to those which occur in the

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Taylor problem. Our present investigations are mainly aimed at finding out how the results for incompressible flows found by Hall (1982), Hall and Lakin (1985), Hall and Seddougui (1989) and Seddougui and Bassom (1990) should be modified in order to describe hypersonic flows.

The most important property of a two dimensional hypersonic boundary layer is probably that it can be divided in the large Mach number limit into a wall layer and a temperature adjustment layer sitting at the edge of the boundary layer, and that it is the latter layer that is most susceptible to Görtler vortices. Since for a hypersonic boundary layer the basic state temperature varies significantly, Chapman's viscosity law is a poor approximation to the viscosity of the fluid and Sutherland's law is more realistic, throughout the present paper Sutherland's law will be adopted and our attention will be focussed on the temperature adjustment layer. It has been shown by Fu, Hall and Blackaby (1990) that when Sutherland's law is used, the wall layer and the temperature adjustment layer are respectively of thickness of order $M^{3/2}$ and order unity in terms of the physical variable. For Görtler vortices which have wavelength comparable with the boundary layer thickness (defined as the *wall layer mode*), the neutral Görtler number is a decreasing function of the local wavenumber. As the local wavenumber increases (physically, this may correspond to when we follow the downstream evolution of the vortices), the centre of vortex activity moves towards the temperature adjustment layer and the neutral Görtler number tends to be independent of the global wavenumber. For Görtler vortices trapped in the temperature adjustment layer, the neutral Görtler number is found to have its first term independent of the global wavenumber. This term is due to the curvature of the basic state; other higher order correction terms are related to viscous effects and are in general affected by boundary layer growth. How important the nonparallel effects are depends upon the wall curvature and on the size of the wavenumber. Before we review Fu, Hall and Blackaby's (1990) main findings, let us first note that for a hypersonic boundary layer described using a similarity variable, boundary layer growth has two lengthscales. The first scale is related to the similarity variable η (which is defined as the ratio of the Howarth-Dorodnitsyn variable over $\sqrt{2x}$ where x is the streamwise variable). This scale, which we shall refer as the *short scale*, is not present in incompressible boundary layers and it arises because of the fact that $\eta_x = O(M^{3/2}) \gg 1$. The second scale is the usual one related to the variable x. It was shown in Fu, Hall and Blackaby (1990) that the effect of boundary growth over the short scale is felt mainly through an $O(M^{3/2})$ curvature term in the y-momentum equation and that when the wall curvature distribution is proportional to $(2x)^{-3/2}$, the wall curvature exactly counterbalances the curvature of the basic state if the Görtler number is chosen appropriately, so that for this special curvature case nonparallel effects affect the stability properties in a similar fashion to that for incompressible boundary layers. But since the special curvature case is possibly of little physical relevance we shall only consider the more general curvature case in this paper.

Intuitively, the effects of boundary layer growth decrease as the wavelength becomes increasingly small, since to a very small wavelength vortex, the boundary layer streamlines would be almost straight lines. Therefore the effects of boundary growth are usually described in terms of the relative order of the vortex wavelength to the lengthscale over which the boundary layer growth is significant (i.e. is an $O(1)$ effect). Since for a hypersonic boundary layer the most important scale of boundary layer growth is the short scale defined above and it is related to the free stream Mach number, we describe nonparallel effects in terms of the relative orders of the wavenumber and Mach number. Fu, Hall and Blackaby (1990) showed that when the local wavenumber is of order unity, the downstream development of Görtler vortices is governed by inviscid equations and thus their spatial development has an oscillatory nature. Viscous effects are small but are cumulative so that they become important further downstream where the local wavenumber has become large. Nonparallel effects are dominant for this range of wavenumbers. When the wavenumber reaches $O(M^{1/4})$, viscous effects become of leading order effects but nonparallel effects are still dominant. Only when the wavenumber becomes as large as of order $M^{3/8}$, do nonparallel effects become negligible and viscous effects then dictate the downstream evolution properties of Görtler vortices.

In the present investigation, our first aim is to find out how nonlinear effects compete with viscous effects in the evolution of Görtler vortices and to show how a large amplitude vortex structure can be established. We exclude nonparallel effects by assuming that the local wavenumber is of order $M^{3/8}$. This assumption is physically relevant because a large local wavenumber can not only be achieved by a large global wavenumber but it can also be achieved by moving sufficiently far downstream. Thus a vortex of any wavenumber would eventually evolve into the wavenumber regime covered by the present theory as they propagate downstream. Our second aim is to provide an asymptotic description for the wavy type of secondary instability which the large amplitude Görtler vortex structure may suffer.

The rest of this paper is divided into five sections as follows. After discussing the basic state and giving a brief review of the linear theory in section 2, we consider in section 3 the weakly nonlinear development of large amplitude Görtler vortices in a small neighbourhood of the neutral stability position given by the linear theory. We show that if the basic state and the wall curvature satisfy a certain condition, these Görtler vortices will grow until they become so large as to drive a mean flow correction as large as the basic state. The further downstream evolution of these large amplitude vortices is then studied in section 4 where we show that the vortices have a triple layer structure which consists of a region of vortex activity bounded by two transition layers over which the amplitude of the harmonic part of the vortex decays to zero exponentially. The position of the two transition layers are found to be governed by a free boundary problem which is solved in section 5 for a number of curvature distribution cases. In section 6 we investigate the secondary instability of the two transition layers with respect to

travelling waves which are $\pi/2$ out of phase in the spanwise direction with the steady Görtler vortices. Finally in section 6 we discuss our results and draw some conclusions.

2   Basic state and a review of the linear theory

Consider a hypersonic boundary layer over a rigid wall of variable curvature $(1/A)\kappa(x^*/L)$, where L is a typical streamwise length scale and A is a lengthscale characterizing the radius of curvature of the wall. We choose a curvilinear coordinate system (x^*, y^*, z^*) with x^* measuring distance along the wall, y^* perpendicular to the wall and z^* in the spanwise direction. The corresponding velocity components are denoted by (u^*, v^*, w^*) and density, temperature and viscosity by ρ^*, T^* and μ^* respectively. The free stream values of these quantities will be signified by a subscript ∞. We define a curvature parameter δ by

$$\delta = \frac{L}{A}, \tag{1}$$

and consider the limit $\delta \to 0$ with the Reynolds number R defined by

$$R = \frac{u_\infty^* L \rho_\infty^*}{\mu_\infty^*} \tag{2}$$

taken to be so large that the Görtler number

$$G = 2R^{1/2}\delta \tag{3}$$

is $O(1)$. In the following analysis, coordinates (x^*, y^*, z^*) are scaled on $(L, R^{-1/2}L, R^{-1/2}L)$, the velocity (u^*, v^*, w^*) is scaled on $(u_\infty^*, R^{-1/2}u_\infty^*, R^{-1/2}u_\infty^*)$ and other quantities such as ρ^*, T^*, and μ^* are scaled on their free stream values with the only exception that the pressure p^* is scaled on $\rho_\infty^* u_\infty^{*2}$ and the bulk viscosity λ^* is scaled on μ_∞^*. All dimensionless quantities will be denoted by the same letters without a superscript $*$. For an ideal gas without dissociation the Navier-Stokes equations are given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_\beta}(\rho v_\beta) = 0, \tag{4}$$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}(\mu \frac{\partial u}{\partial y}) + \frac{\partial}{\partial z}(\mu \frac{\partial u}{\partial z}), \tag{5}$$

$$\rho(\frac{Dv}{Dt} + \frac{1}{2}G\kappa u^2) = -R\frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left\{(\lambda - \frac{2}{3}\mu)\frac{\partial v_\beta}{\partial x_\beta}\right\} + \frac{\partial}{\partial x_\beta}(\mu \frac{\partial v_\beta}{\partial y})$$
$$+ \frac{\partial}{\partial y}(\mu \frac{\partial v}{\partial y}) + \frac{\partial}{\partial z}(\mu \frac{\partial v}{\partial z}), \tag{6}$$

$$\rho \frac{Dw}{Dt} = -R\frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left\{(\lambda - \frac{2}{3}\mu)\frac{\partial v_\beta}{\partial x_\beta}\right\} + \frac{\partial}{\partial x_\beta}(\mu \frac{\partial v_\beta}{\partial z}) + \frac{\partial}{\partial y}(\mu \frac{\partial w}{\partial y}) + \frac{\partial}{\partial z}(\mu \frac{\partial w}{\partial z}), \tag{7}$$

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$$\rho \frac{DT}{Dt} = \mu(\gamma - 1)M^2[(\frac{\partial u}{\partial y})^2 + (\frac{\partial u}{\partial z})^2] + (\gamma - 1)M^2 \frac{Dp}{Dt}$$

$$+ \frac{1}{\sigma} \frac{\partial}{\partial y}(\mu \frac{\partial T}{\partial y}) + \frac{1}{\sigma} \frac{\partial}{\partial z}(\mu \frac{\partial T}{\partial z}), \tag{8}$$

$$\gamma M^2 p = \rho T. \tag{9}$$

Here we have used a mixed notation in which (v_1, v_2, v_3) is identified with (u, v, w) and (x_1, x_2, x_3) with (x, y, z). Repeated suffices β signify summation from 1 to 3. The constants γ, M and σ are in turn the ratio of specific heats, the free stream Mach number and the Prandtl number defined by

$$\gamma = \frac{c_{v\infty}}{c_{p\infty}}, \quad M = \frac{u_\infty^{*2}}{\gamma \tilde{\Re} T_\infty^*} = \frac{u_\infty^{*2}}{a_\infty^2}, \quad \sigma = \frac{\mu_\infty c_{p\infty}}{k_\infty},$$

where $\tilde{\Re}$ is the gas constant, k is the coefficient of heat conduction, and $a_\infty = \sqrt{\gamma \tilde{\Re} T_\infty^*}$ is the sound speed in the free stream. In equations (5)–(8), the operator D/Dt is the material derivative and it has the usual expression appropriate to a rectangular coordinate system.

The basic state is given by

$$(u, v, w) = (\bar{u}(x,y), \bar{v}(x,y), 0), \quad T = \bar{T}(x,y),$$

$$\rho = \bar{\rho}(x,y), \quad \mu = \bar{\mu}(x,y). \tag{10}$$

By substituting (10) into the governing equations (4)–(9) it is straightforward to obtain the reduced equations satisfied by the basic state. The reader is referred to the book by Stewartson (1964) for a detailed discussion of these basic state equations. If we define the Howarth-Dorodnitsyn variable $\bar{y}$ and a similarity variable η by

$$\bar{y} = \int_0^y \bar{\rho} dy \quad \text{and} \quad \eta = \frac{\bar{y}}{\sqrt{2x}}, \tag{11}$$

then the continuity equation is satisfied if $\bar{u}$ and $\bar{v}$ are written as

$$\bar{u} = f'(\eta), \quad \bar{v} = \frac{1}{\sqrt{2x}}[-\frac{1}{\bar{\rho}}f(\eta) + f'(\eta) \int_0^\eta \frac{1}{\bar{\rho}}d\eta]. \tag{12}$$

Here the functions $f(\eta)$ and $\bar{T}(\eta)$ must satisfy

$$ff'' + (\bar{\rho}\bar{\mu}f'')' = 0, \tag{13}$$

$$\frac{1}{\sigma}(\bar{\rho}\bar{\mu}\bar{T}')' + f\bar{T}' + \bar{\mu}(\gamma - 1)M^2 \bar{\rho}(f'')^2 = 0, \tag{14}$$

if the x-momentum and energy equations are to be satisfied. The y-momentum equation gives

$$\frac{\partial \bar{p}}{\partial y} = 0$$

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to leading order so that $\bar{p} = \bar{p}(x)$. In our following analysis, we assume that there is no pressure gradient along the streamwise direction and therefore we can take $\bar{p} = $ constant. Equation (9) then gives

$$\bar{\rho}\bar{T} = 1. \tag{15}$$

Once the viscosity $\bar{\mu}$ is specified as a function of the temperature, equations (13) and (14) can then be integrated to determine the basic state. Such solutions have been given by Hall and Fu (1989) for Chapman's viscosity law and by Fu, Hall and Blackaby (1990) for Sutherland's law. In both cases, the boundary layer divides into two sublayers: a wall layer in which $\bar{T} = O(M^2)$ and a temperature adjustment layer over which the temperature decreases monotonically to its free-stream value. Certainly, for a hypersonic boundary layer across which the temperature varies significantly, it is more appropriate to use Sutherland's viscosity law

$$\bar{\mu} = (1 + m)\frac{\bar{T}^{3/2}}{(\bar{T} + m)} \tag{16}$$

where m is a constant. In this case, the thicknesses of the wall layer and the adjustment layer are $O(M^{-1/2})$ and $O(1)$, respectively (in terms of the similarity variable η). In the adjustment layer, the functions f and $\bar{T}$ in (12) expand as

$$f = \eta - \frac{\beta}{M^{1/2}} + \frac{\hat{f}(\eta)}{M^{\frac{3}{2\sigma}+\frac{1}{2}}} + \cdots, \quad \bar{T} = \hat{T} + \cdots, \tag{17}$$

with $\hat{f}$ and $\hat{T}$ satisfying

$$(1 + m)\left(\frac{\sqrt{\hat{T}}}{\hat{T} + m}\hat{f}''\right)' + \eta\hat{f}'' = 0, \quad \frac{(1 + m)}{\sigma}\left(\frac{\sqrt{\hat{T}}}{\hat{T} + m}\hat{T}'\right)' + \eta\hat{T}' = 0. \tag{18}$$

In the linear stability analysis, we superimpose a steady periodic stationary vortex structure with wavenumber a on the basic state (10) and the perturbation equations are found by linearizing the Navier-Stokes equations about the basic state. These linear equations have been fully discussed in our previous paper Fu, Hall and Blackaby (1990). It was shown there that for the wall mode which has wavelength comparable with the boundary layer thickness, the neutral Görtler number is a decreasing function of the local wavenumber. As the latter increases, the centre of vortex activity moves towards the temperature adjustment layer and the Görtler number tends to a constant which is the leading order term of the Görtler number expansion for the mode trapped in the temperature adjustment layer. It is this mode that is most susceptible to Görtler vortices since it has a smaller Görtler number than any other mode.

As is typical of Görtler vortices in growing boundary layers, the evolution of Görtler vortices in the temperature adjustment layer is dominated by nonparallel effects. It was shown in Fu, Hall and Blackaby (1990) that in the hypersonic limit such nonparallel effects operate mainly

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through the $O(M^{3/2})$ curvature of the basic state. Thus only when the wavenumber a is as large as of order $M^{3/8}$ do nonparallel effects become negligible and the following asymptotic expression for the neutral Görtler number can be obtained:

$$G = \frac{2BM^{3/2}}{\kappa(x_n)(2x_n)^{3/2}} + g_0 a^4 + a^3 \cdot \frac{1}{\sqrt{2x_n}}\sqrt{\frac{3g_0}{2\hat{T}_0^2}\frac{\partial^2 g_0}{\partial \eta^{*2}}} + \cdots,$$

(19)

where

$$B \stackrel{\text{def.}}{=} \lim_{M\to\infty} M^{-3/2} \int_0^\infty \bar{T}(\eta)d\eta, \quad g_0 = -\frac{2\sqrt{2x_n}\bar{\mu}_0^2\hat{T}_0^4}{\sigma\kappa(x_n)\hat{T}_1};$$

(20)

and $\hat{T}_0 = \hat{T}(\eta^*)$, $\hat{T}_1 = \hat{T}'(\eta^*)$, $\bar{\mu}_0 = \bar{\mu}(\hat{T}_0)$. The constant η^* denotes the centre of vortex activity and has the numerical value of 3.001 when $\sigma = 0.72$, $m = 0.509$. In (19) the first term is due to the curvature of the basic state and other terms are due to viscous effects. It is clear that $a = O(M^{3/8})$ is the order at which viscous effects become comparable with the effects of centrifugal acceleration due to the curvature of the basic state. In the following analysis, our nonlinear theories will be concerned with wavenumbers in this regime. Although this assumption about the order of the wavenumber is necessary to obtain a relatively simple asymptotic analysis, it is also relevant to physically situations since our analysis is actually based on the assumption $\sqrt{2x}a = O(M^{3/8})$ and the latter condition is always satisfied by vortices far downstream of the leading edge.

3 Weakly nonlinear theory

In this section, we consider the initial nonlinear evolution of large wavenumber Görtler vortices in the neighbourhood of the neutral position $x = x_n$. We shall fix the Görtler number as given by

$$G = \left(\frac{2BN}{\kappa(x_n)(2x_n)^{3/2}} + g_0\right)\frac{1}{\epsilon^4},$$

(1)

where for the convenience of asymptotic analysis, we have defined an $O(1)$ constant N and a small parameter ϵ by

$$N = M^{\frac{3}{2}} \cdot a^{-4}, \quad \epsilon = 1/a.$$

(2)

It can easily be seen that (1) is just (19) with the $O(a^3)$ term on the right hand side neglected. This implies that the linear neutral position corresponding to (1) would be $x_n + O(\epsilon)$. In the present context, it does not matter whether we start with (19) or with (1).

We now turn to the derivation of the nonlinear perturbation equations. The total flow is written as

$$u = \bar{u} + \frac{1}{M_1}U, \quad v = \bar{v} + V, \quad w = W,$$

$$p = \bar{p} + \frac{1}{R}(\tilde{p} + P), \quad T = \bar{T} + T,$$

(3)

where M_1 defined by

$$M_1 = \frac{1}{M^{\frac{3}{2\sigma}+\frac{1}{2}}}$$

is used to scale the streamwise perturbation velocity so that M will not appear in the following nonlinear perturbation equations. In (3d), $\tilde{p}/R$ is the second order correction to the basic state pressure (the leading order term $\bar{p}$ is a constant). On substituting (3) into the Navier-Stokes equations (4)–(9), making use of (12) and (17) and neglecting cubic and higher order terms of the perturbation quantities U, V, W, P and T, we obtain the following set of perturbation equations:

$$\frac{1}{\hat{T}}\left(\frac{\partial U}{\partial x} - \frac{\eta}{2x}\frac{\partial U}{\partial \eta}\right) - \bar{\mu}U_{zz} - \frac{1}{2x\hat{T}}\frac{\partial}{\partial \eta}\left(\frac{\bar{\mu}}{\hat{T}}\frac{\partial U}{\partial \eta}\right) + \frac{\hat{f}''}{\sqrt{2x}\hat{T}^2}V$$

$$+[\frac{\eta \hat{f}''}{2x\hat{T}^2} - \frac{1}{2x\hat{T}}\frac{\partial}{\partial \eta}(\frac{\bar{\mu}}{\hat{T}}\hat{f}'')]T - \frac{\bar{\mu}\hat{f}''}{2x\hat{T}^2}\frac{\partial T}{\partial \eta} + \frac{1}{\hat{T}}(\frac{1}{\sqrt{2x}\hat{T}}V\frac{\partial U}{\partial \eta} + WU_z)$$

$$-\bar{\mu}(TU_z)_z - \frac{T}{\hat{T}^2}[\frac{\partial U}{\partial x} - \frac{\eta}{2x}\frac{\partial U}{\partial \eta} + \frac{\hat{f}''}{\sqrt{2x}\hat{T}}V] - \frac{1}{2x\hat{T}}\frac{\partial}{\partial \eta}(\frac{\bar{\mu}}{\hat{T}}T\frac{\partial U}{\partial \eta})$$

$$-\frac{\eta \hat{f}''}{2x}\frac{T^2}{\hat{T}^3} - \frac{1}{4x\hat{T}}\frac{\partial}{\partial \eta}(\frac{\tilde{\mu}\hat{f}''}{\hat{T}}T^2) = 0, \tag{4}$$

$$\frac{1}{\hat{T}}\left(\frac{\partial V}{\partial x} - \frac{\eta}{2x}\frac{\partial V}{\partial \eta}\right) - \frac{\eta \hat{T}'}{2x\hat{T}^2}V - \bar{\mu}V_{zz} - \frac{4}{3\cdot(2x)\hat{T}}\frac{\partial}{\partial \eta}(\frac{\bar{\mu}}{\hat{T}}\frac{\partial V}{\partial \eta})$$

$$+\frac{1}{\sqrt{2x}\hat{T}}\frac{\partial P}{\partial \eta} - \left\{\frac{1}{\hat{T}^2(2x)^{3/2}}[\frac{1}{2}\kappa\cdot(2x)^{3/2}G - BM^{3/2} - \frac{\bar{\mu}\hat{T}'}{\sigma\hat{T}} + \eta^2\hat{T}']\right.$$

$$\left. -\frac{4}{3(2x)^{3/2}\hat{T}}\frac{\partial}{\partial \eta}(\frac{\eta \hat{T}'\bar{\mu}}{\hat{T}})\right\}T + \frac{4\bar{\mu}\eta\hat{T}'}{3(2x)^{3/2}\hat{T}^2}\frac{\partial T}{\partial \eta}$$

$$+\frac{2\tilde{\mu}\hat{T}'}{3\sqrt{2x}\hat{T}}W_z - \frac{\bar{\mu}}{3\sqrt{2x}\hat{T}}\frac{\partial}{\partial \eta}(W_z) + \frac{1}{\hat{T}}(\frac{1}{\sqrt{2x}\hat{T}}V\frac{\partial V}{\partial \eta} + WV_z)$$

$$-\frac{T}{\hat{T}^2}(\frac{\partial V}{\partial x} - \frac{\eta}{2x}\frac{\partial V}{\partial \eta} - \frac{\eta \hat{T}'}{2x\hat{T}}V) + [\frac{2}{3}\tilde{\mu}W_z - \frac{4\tilde{\mu}}{3\sqrt{2x}\hat{T}}\frac{\partial V}{\partial \eta}]\cdot\frac{1}{\sqrt{2x}\hat{T}}\frac{\partial T}{\partial \eta}$$

$$-\left\{\frac{4}{3\cdot(2x)\hat{T}^2}\frac{\partial \tilde{\mu}}{\partial \eta}\frac{\partial V}{\partial \eta} + \frac{4\tilde{\mu}}{3\cdot(2x)\hat{T}}\frac{\partial}{\partial \eta}(\frac{1}{\hat{T}}\frac{\partial V}{\partial \eta}) + \frac{\tilde{\mu}}{3\sqrt{2x}\hat{T}}\frac{\partial W_z}{\partial \eta}\right.$$

$$\left. -\frac{2}{3\sqrt{2x}\hat{T}}\frac{\partial \tilde{\mu}}{\partial \eta}W_z + \tilde{\mu}V_{zz}\right\}T - [\frac{1}{\sqrt{2x}\hat{T}}\frac{\partial W}{\partial \eta} + V_z]\tilde{\mu}T_z$$

$$+\frac{T^2}{\hat{T}^3(2x)^{3/2}}[\frac{1}{2}\kappa\cdot(2x)^{3/2}G - BM^{3/2} - \frac{\bar{\mu}\hat{T}'}{\sigma\hat{T}} + \eta^2\hat{T}'] + \frac{1}{6(2x)^{3/2}\hat{T}}\frac{\partial}{\partial \eta}\left(\frac{\tilde{\mu}\eta\hat{T}'}{\hat{T}}T^2\right) = 0, \tag{5}$$

$$\frac{\bar{\mu}\hat{T}'}{\sqrt{2x}\hat{T}}V_z + \frac{\bar{\mu}}{3\sqrt{2x}\hat{T}}\frac{\partial V_z}{\partial \eta} - P_z + \frac{2\bar{\mu}\eta\hat{T}'}{3\cdot(2x)\hat{T}}T_z - \frac{1}{\hat{T}}(\frac{\partial W}{\partial x} - \frac{\eta}{2x}\frac{\partial W}{\partial \eta})$$

$$+\frac{4}{3}\bar{\mu}W_{zz} + \frac{1}{2x\hat{T}}\frac{\partial}{\partial \eta}(\frac{\bar{\mu}}{\hat{T}}\frac{\partial W}{\partial \eta}) - \frac{1}{\hat{T}}(\frac{1}{\sqrt{2x}\hat{T}}V\frac{\partial W}{\partial \eta} + WW_z)$$

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$$+\frac{T}{\hat{T}^2}(\frac{\partial W}{\partial x} - \frac{\eta}{2x}\frac{\partial W}{\partial \eta}) + \{\frac{4}{3}\bar{\mu}W_{zz} + \frac{\bar{\mu}}{3\sqrt{2x\hat{T}}}\frac{\partial V_z}{\partial \eta} + \frac{1}{\sqrt{2x\hat{T}}}\frac{\partial \bar{\mu}}{\partial \eta}V_z$$

$$+\frac{1}{2x\hat{T}^2}\frac{\partial \bar{\mu}}{\partial \eta}\frac{\partial W}{\partial \eta} + \frac{\bar{\mu}}{2x\hat{T}}\frac{\partial}{\partial \eta}(\frac{1}{\hat{T}}\frac{\partial W}{\partial \eta})\}T + \frac{\bar{\mu}}{\sqrt{2x\hat{T}}}(\frac{1}{\sqrt{2x\hat{T}}}\frac{\partial W}{\partial \eta} + V_z)\frac{\partial T}{\partial \eta}$$

$$+(\frac{4}{3}W_z - \frac{2}{3\sqrt{2x\hat{T}}}\frac{\partial V}{\partial \eta})\tilde{\mu}T_z = 0, \qquad (6)$$

$$\frac{1}{\sqrt{2x\hat{T}}}\frac{\partial}{\partial \eta}(\frac{V}{T}) + \frac{1}{\hat{T}}W_z + \frac{\eta\hat{T}'}{2x\hat{T}^3}\cdot T - (\frac{\partial}{\partial x} - \frac{\eta}{2x}\frac{\partial}{\partial \eta})(\frac{T}{\hat{T}^2})$$

$$-(\frac{1}{\sqrt{2x\hat{T}}}\frac{\partial V}{\partial \eta} + W_z)\frac{T}{\hat{T}^2} - (\frac{1}{\sqrt{2x\hat{T}}}V\frac{\partial}{\partial \eta} + W\frac{\partial}{\partial z})(\frac{T}{\hat{T}^2})$$

$$+(\frac{\partial}{\partial x} - \frac{\eta}{2x}\frac{\partial}{\partial \eta})(\frac{T^2}{\hat{T}^3}) - \frac{\eta\hat{T}'}{2x\hat{T}^4}T^2 = 0, \qquad (7)$$

$$\frac{\hat{T}'}{\sqrt{2x}\hat{T}^2}V + \frac{1}{\hat{T}}(\frac{\partial T}{\partial x} - \frac{\eta}{2x}\frac{\partial T}{\partial \eta}) + [\frac{\eta\hat{T}'}{2x\hat{T}^2} - \frac{1}{2x\sigma\hat{T}}\frac{\partial}{\partial \eta}(\frac{\bar{\mu}\hat{T}'}{\hat{T}})]T - \frac{\bar{\mu}}{\sigma}T_{zz}$$

$$-\frac{\tilde{\mu}\hat{T}'}{2x\hat{T}^2\sigma}\frac{\partial T}{\partial \eta} - \frac{1}{2x\sigma\hat{T}}\frac{\partial}{\partial \eta}(\frac{\bar{\mu}}{\hat{T}}\frac{\partial T}{\partial \eta}) + \frac{1}{\hat{T}}(\frac{1}{\sqrt{2x\hat{T}}}V\frac{\partial T}{\partial \eta} + WT_z)$$

$$-\frac{T}{\hat{T}^2}(\frac{\partial T}{\partial x} - \frac{\eta}{2x}\frac{\partial T}{\partial \eta} + \frac{\hat{T}'}{\sqrt{2x}\hat{T}}V) - \frac{1}{2x\sigma\hat{T}^2}\frac{\partial \tilde{\mu}}{\partial \eta}T\frac{\partial T}{\partial \eta} - \frac{\eta\hat{T}'}{2x\hat{T}^3}T^2$$

$$-\frac{\bar{\mu}}{\sigma}[\frac{1}{2x\hat{T}}\frac{\partial}{\partial \eta}(\frac{T}{\hat{T}}\frac{\partial T}{\partial \eta}) + (TT_z)_z] - \frac{1}{4x\sigma\hat{T}}\frac{\partial}{\partial \eta}\left(\frac{\tilde{\mu}\hat{T}'}{\hat{T}}T^2\right) = 0. \qquad (8)$$

Here $\bar{\mu} = \mu(\hat{T}), \tilde{\mu} = d\bar{\mu}/d\hat{T}, \tilde{\tilde{\mu}} = d^2\bar{\mu}/d\hat{T}^2$ and the bulk viscosity has been taken to be zero. The linear perturbation equations discussed in Fu, Hall and Blackaby (1990) can be obtained from the above equations by neglecting nonlinear terms.

To solve these nonlinear perturbation equations, we first note that in the large wavenumber limit, Görtler vortices are trapped in an internal viscous layer of $O(\epsilon^{1/2})$ thickness centred at the most unstable position $\eta = \eta*$. We therefore define a variable ϕ by

$$\phi = \frac{\eta - \eta^*}{\epsilon^{1/2}}. \qquad (9)$$

In the neighbourhood of the neutral position $x = x_n$, the growth rate in the x-direction can be shown to be of order $1/\epsilon$ and it is appropriate to describe such rapid growth by defining another variable $\tilde{x}$ by

$$\tilde{x} = \frac{x - x_n}{\epsilon}. \qquad (10)$$

An order of magnitude analysis of the perturbation equations (4)–(8) then shows that

$$U = O(\epsilon^2 V), \quad W = O(\epsilon^{1/2}V), \quad T = O(\epsilon^2 V), \quad P = O(\epsilon^{-1/2}V). \qquad (11)$$

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We assume that nonlinear effects reinforce the fundamental at the order in ϵ at which the vertical structure is determined. It can then be deduced that the appropriate sizes of the perturbation quantities must be

$$U = O(\epsilon^{3/2}), \quad V = O(\epsilon^{-1/2}), \quad W = O(1)$$

$$T = (\epsilon^{3/2}), \quad P = O(\epsilon^{-1}), \tag{12}$$

and that the corresponding mean flow corrections, signified by a subscript "m", must be of the orders

$$U_m = O(\epsilon^{3/2}), \quad V_m = O(\epsilon), \quad P_m = O(\epsilon^{-2}), \quad T_m = O(\epsilon^{3/2}). \tag{13}$$

As is well known, the nonlinear interactions that occur in the Taylor-Görtler problem do not generate a mean flow in the spanwise direction. We therefore look for asymptotic solutions of the form

$$U = \epsilon^{3/2} \left\{ u_{m0} + \epsilon^{1/2} u_{m1} + \cdots + [(U_0 + \epsilon^{1/2} U_1 + \cdots)E + \cdots + C.C.] \right\},$$

$$V = \epsilon^{-1/2} \left\{ (V_0 + \epsilon^{1/2} V_1 + \cdots)E + \cdots + C.C. \right\},$$

$$W = (W_0 + \epsilon^{1/2} W_1 + \cdots)E + \cdots + C.C., \tag{14}$$

$$T = \epsilon^{3/2} \left\{ \theta_{m0} + \epsilon^{1/2} \theta_{m1} + \cdots + [(\theta_0 + \epsilon^{1/2} \theta_1 + \cdots)E + \cdots + C.C.] \right\},$$

$$P = \epsilon^{-2}(P_{m0} + \epsilon^{1/2} P_{m1} + \cdots) + \epsilon^{-1} \left\{ (P_0 + \epsilon^{1/2} P_1 + \cdots)E + \cdots + C.C. \right\},$$

where

$$E = \exp(\frac{iz}{\epsilon}), \tag{15}$$

and C.C. denotes the conjugate. On substituting these expansions into the perturbation equations (4)–(8), expanding all coefficients there about $x = x_n$ and $\eta = \eta^*$, and then equating the coefficients of like powers such as $\epsilon, E\epsilon$ and etc., we obtain an infinite hierarchy of equations. To leading order, the Görtler number g_0 is determined from a solvability condition for (V_0, T_0) and is given by

$$g_0 = -\frac{2\sqrt{2x_n}\bar{\mu}_0^2 \hat{T}_0^4}{\sigma \kappa_0 \hat{T}_1}, \tag{16}$$

whilst U_0, W_0, θ_0 and P_0 are related to V_0 by

$$U_0 = -\frac{\hat{f}_0''}{\sqrt{2x_n}\bar{\mu}_0 \hat{T}_0^2} V_0, \qquad \theta_0 = -\frac{\sigma \hat{T}_1}{\sqrt{2x_n}\bar{\mu}_0 \hat{T}_0^2} V_0,$$

$$iW_0 = -\frac{1}{\sqrt{2x_n}\hat{T}_0} \frac{\partial V_0}{\partial \phi}, \qquad P_0 = -\frac{\bar{\mu}_0}{\sqrt{2x_n}\hat{T}_0} \frac{\partial V_0}{\partial \phi}, \tag{17}$$

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where $\hat{f}_0'' = \hat{f}''(\eta^*)$, $\hat{T}_0 = \hat{T}(\eta^*)$, $\hat{T}_1 = \hat{T}'(\eta^*)$, $\bar{\mu}_0 = \bar{\mu}(\hat{T}_0)$ $\kappa_0 = \kappa(x_n)$. Note that (16) is consistent with (20), as we would expect. The mean pressure, the mean streamwise velocity and the mean temperature are found to satisfy

$$\frac{\partial P_{m0}}{\partial \phi} = \frac{\kappa_0 \sqrt{2x_n} g_0}{2\hat{T}_0} \theta_{m0}, \tag{18}$$

$$\left(\frac{2x_n \sigma \hat{T}_0}{\bar{\mu}_0} \frac{\partial}{\partial \tilde{x}} - \frac{\partial^2}{\partial \phi^2} \right) \theta_{m0} = \frac{2\sigma^2 \hat{T}_1}{\bar{\mu}_0^2 \hat{T}_0^2} \frac{\partial |\bar{V}_0|^2}{\partial \phi}, \tag{19}$$

and

$$\left(\frac{2x_n \hat{T}_0}{\bar{\mu}_0} \frac{\partial}{\partial \tilde{x}} - \frac{\partial^2}{\partial \phi^2} \right) u_{m0} = \frac{2\hat{f}_0''}{\bar{\mu}_0^2 \hat{T}_0^2} \frac{\partial |\bar{V}_0|^2}{\partial \phi}. \tag{20}$$

To next order, we obtain four expressions similar to (17) for U_1, θ_1, W_1 and P_1 in terms of V_1 and V_0 and the condition that

$$\frac{dg_0}{d\eta}|_{\eta=\eta^*} = 0, \tag{21}$$

which implies that η^* is where g_0 attains its minimum (and hence that η^* is the most unstable position).

If we carry out the expansion to one order higher, we find from a solvability condition for (V_2, θ_2) that V_0 must satisfy the evolution equation

$$\frac{\partial^2 V_0}{\partial \phi^2} - \frac{2(1+\sigma)\hat{T}_0 x_n}{3\bar{\mu}_0} \frac{\partial V_0}{\partial \tilde{x}} - \tilde{a}\phi^2 V_0 + \tilde{b}\tilde{x}V_0 = -\frac{2x_n \hat{T}_0^2}{3\hat{T}_1} V_0 \frac{\partial \theta_{m0}}{\partial \phi}, \tag{22}$$

where

$$\tilde{a} = \frac{\hat{T}_0^2 x_n}{3g_0} \frac{\partial^2 g_0}{\partial \eta^2}|_{\eta=\eta^*} > 0,$$

$$\tilde{b} = \frac{2x_n \hat{T}_0^2}{3} \left\{ \frac{2BN}{g_0 \kappa_0 (2x_n)^{3/2}} \left(\frac{\kappa_1}{\kappa_0} + \frac{3}{2x_n} \right) + \frac{\kappa_1}{\kappa_0} - \frac{1}{2x_n} \right\}, \tag{23}$$

and where $\kappa_1 = \kappa'(x_n)$. Thus we have two coupled evolution equations (19) and (22) together with an uncoupled equation (20) which govern the downstream evolution of the fundamentals and the mean flow corrections. It is easily seen that the present weakly nonlinear theory is different from the classical weakly nonlinear theory (see, for example, Stuart (1965)). In the latter theory the amplitudes of the mean flow corrections are one order smaller than those of the fundamentals and thus the resulting evolution equation for the fundamentals is an ordinary differential equation and the former can be determined independently of the mean flow corrections.

Before we present the solutions of these nonlinear evolution equations, let us first note that equation (22) reduces to the linear evolution equation

$$\frac{\partial^2 V_0}{\partial \phi^2} - \frac{2(1+\sigma)\hat{T}_0 x_n}{3\bar{\mu}_0} \frac{\partial V_0}{\partial \tilde{x}} - \tilde{a}\phi^2 V_0 + \tilde{b}\tilde{x}V_0 = 0, \tag{24}$$

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after the nonlinear forcing term on the right hand side has been neglected. This equation has been discussed in Fu, Hall and Blackaby (1990) and its general solution can be obtained by first looking for separable solutions and then expanding in terms of the eigenfunctions. It is shown there that the mth mode is neutrally stable at $\tilde{x}_n = (2m + 1)\sqrt{\tilde{a}}/\tilde{b}$. The most unstable mode corresponds to $m = 0$. Therefore Görtler vortices with Görtler number given by (1) are neutrally stable at $x_n + \epsilon \tilde{x}_n$. If we replace x_n in (1) by $x_n - \epsilon \tilde{x}_n$ and expand the resulting expression up to and include the $O(\epsilon^{-3})$ term, we recover (19) which is the appropriate Görtler number expansion for Görtler vortices neutrally stable at $x = x_n$. Thus, as we remarked in the paragraph below (2), it does not matter whether we use (19) or (1) for our weakly nonlinear theory; such a difference in the choice of the Görtler number only results in an $O(\epsilon)$ shift in the linear neutral position.

We now discuss the solutions of the nonlinear evolution equations (19) and (22). We shall not consider (20) any further since its solution for the mean streamwise velocity u_{m0} is not needed in the remaining discussions of this paper. To simplify the notation, we make the following substitutions:

$$X = \frac{|\tilde{b}|}{2\sqrt{\tilde{a}}}\tilde{x}, \qquad \zeta = (4\tilde{a})^{1/4}\phi,$$

$$\tilde{V}_0 = \frac{\sigma}{\bar{\mu}_0(4\tilde{a})^{1/4}}\sqrt{\frac{2x_n}{3}} \cdot V_0, \qquad \tilde{\theta}_{m0} = -\frac{2x_n \hat{T}_0^2}{6\hat{T}_1(4\tilde{a})^{1/4}}\theta_{m0}. \tag{25}$$

Equations (19) and (22) then become

$$\left(\frac{\partial^2}{\partial \zeta^2} - \frac{2\sigma}{1+\sigma}k\frac{\partial}{\partial X}\right)\tilde{\theta}_{m0} = \frac{\partial|\tilde{V}_0|^2}{\partial \zeta}, \tag{26}$$

$$\left(\frac{\partial^2}{\partial \zeta^2} - \frac{2}{3}k\frac{\partial}{\partial X} - \frac{1}{4}\zeta^2 \pm X\right)\tilde{V}_0 = 2\tilde{V}_0\frac{\partial\tilde{\theta}_{m0}}{\partial \zeta}, \tag{27}$$

where

$$k = \frac{(1+\sigma)x_n\hat{T}_0|\tilde{b}|}{4\bar{\mu}_0\tilde{a}}.$$

In (27), the positive (negative) sign is to be taken if $\tilde{b}$ is positive (negative). These two equations are of the same form as Hall's (1982b) equations (3.15a,b) for incompressible flows (they are identical when $\sigma = 1$). The reader is referred to that paper for a detailed discussion of their numerical and asymptotic solutions. According to Hall (1982), an important property of these two coupled evolution equations is that any initial disturbance introduced upstream would either decay to zero or evolve into a *unique* large amplitude structure at large downstream locations, depending on whether $\tilde{b} < 0$ or $\tilde{b} > 0$. The latter conditions are in fact the conditions for Görtler vortices to decay ($\tilde{b} < 0$) or grow ($\tilde{b} > 0$) linearly downstream of the neutral position $x_n + \epsilon \tilde{x}_n$ (see Fu, Hall and Blackaby (1990)).

The easiest way to see how a large amplitude structure is possible is by looking for such solutions for (26) and (27) directly. If an asymptotic state is to be achieved, it must be the last two linear terms on the left hand side of (27) that balance with the nonlinear term on the right hand side of (27). Thus after integration we have

$$\tilde{\theta}_{m0} = \frac{1}{2} X^{3/2} \left\{ \frac{\zeta}{\sqrt{X}} - \frac{1}{12} (\frac{\zeta}{\sqrt{X}})^3 \right\}, \tag{28}$$

where, based on the argument given in Hall (1982), we have assumed $\tilde{\theta}_{m0}$ to be an odd function of ζ so that we can put the arbitrary integration constant to zero. On substituting (28) into (26) and integrating, we obtain

$$|\tilde{V}_0|^2 = \frac{1}{2} (\frac{1}{4} + \frac{\sigma k}{1 + \sigma}) \cdot X \cdot \left\{ C^2 - (\frac{\zeta}{\sqrt{X}})^2 \right\}, \tag{29}$$

where C is an integration constant to be determined. Solutions (28) and (29) are only the first order approximations. We note that the similarity variable $\zeta / X^{1/2}$ is important. Thus if we define

$$\xi = \frac{\zeta}{\sqrt{X}} \tag{30}$$

and look for the following form of asymptotic solutions for (26) and (27):

$$\tilde{\theta}_{m0} = X^{3/2} \tilde{\theta}_{m00}(\xi) + X^{1/2} \tilde{\theta}_{m01}(\xi) + \cdots,$$

$$\tilde{V}_0 = X^{1/2} \tilde{V}_{00}(\xi) + X^{-1/2} \tilde{V}_{01}(\xi) + \cdots, \tag{31}$$

we would have $\tilde{\theta}_{m00}$ and $\tilde{V}_{00}$ given by

$$\tilde{\theta}_{m00} = \frac{1}{2} \left(\xi - \frac{1}{12} \xi^3 \right), \quad \tilde{V}_{00}^2 = \frac{1}{2} (\frac{1}{4} + \frac{\sigma k}{1 + \sigma}) \cdot \left(C^2 - \xi^2 \right). \tag{32}$$

Here, without loss of generality, we have assumed $\tilde{V}_0$ to be real.

The above solution breaks down near $\xi = \pm C$ since $\tilde{V}_{00}^2$ must be positive. In each of these regions $\tilde{V}_{00}$ develops a boundary layer structure (hereafter we shall call them transition layers) and it can be shown that each of the required layer is of thickness $X^{-1/6}$. In the upper transition layer $\xi = C$ we define a new variable ψ by

$$\psi = X^{1/6}(CX^{1/2} - \zeta) = X^{2/3}(C - \xi). \tag{33}$$

To match with the core region solutions (31) and (32), we have to look for solutions of the form

$$\tilde{\theta}_{m0} = X^{3/2} \tilde{\theta}_{m0}^0(\psi) + X^{5/6} \tilde{\theta}_{m0}^1(\psi) + X^{1/6} \tilde{\theta}_{m0}^2 + \cdots,$$

$$\tilde{V}_0 = X^{1/6} \tilde{V}_0^0(\psi) + X^{-1/2} \tilde{V}_0^1(\psi) + \cdots. \tag{34}$$

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On substituting (34) into (26) and (27), equating the coefficients of like powers of X and then considering the matching conditions at various orders, we find that $\tilde{\theta}_{m0}^0$ and $\tilde{\theta}_{m0}^1$ are simply the expansions of the core region solution (32a) in the transition layer, i.e.,

$$\tilde{\theta}_{m0}^0 = \frac{1}{2}(C - \frac{1}{12}C^3), \quad \tilde{\theta}_{m0}^1 = (-\frac{1}{2} + \frac{C^2}{8})\psi, \tag{35}$$

and that $\tilde{V}_0^0$ satisfies

$$-\frac{d^2\tilde{V}_0^0}{d\psi^2} = C(\frac{1}{2} + \frac{2\sigma}{1+\sigma}k)\psi\tilde{V}_0^0 - 2D\tilde{V}_0^0 - 2(\tilde{V}_0^0)^3. \tag{36}$$

By matching with (32b), we require that

$$(\tilde{V}_0^0)^2 \to C(\frac{1}{4} + \frac{\sigma k}{1+\sigma})\psi, \quad \text{as } \psi \to \infty. \tag{37}$$

Equation (36) is a particular form of the second Painlevé transcendent. The existence and uniqueness of its solution has been proved by Hastings and Mcleod (1980), and this solution is proportional to $Ai(-\psi)$ when $\psi \to -\infty$. Thus $\tilde{V}_0$ decays to zero exponentially in the two transition layers which bound the core region of vortex activity.

Above the upper transition layer and below the lower transition layer, (26) reduces to

$$\left(\frac{\partial^2}{\partial\zeta^2} - \frac{2\sigma}{1+\sigma}k\frac{\partial}{\partial X}\right)\tilde{\theta}_{m0} = 0. \tag{38}$$

It admits an asymptotic solution of the form

$$\tilde{\theta}_{m0} = X^{3/2}\Theta_{m0}(\xi) + \cdots. \tag{39}$$

To match with (34a) and (35), $\Theta_{m0}(\xi)$ must satisfy

$$\Theta_{m0} \to \frac{1}{2}(C - \frac{1}{12}C^3), \quad \frac{d\Theta_{m0}}{d\xi} \to \frac{1}{2} - \frac{C^2}{8}, \quad \text{as } \xi \to C. \tag{40}$$

Substituting (39) into (38) gives

$$\left(\frac{d^2}{d\xi^2} + \frac{\sigma k}{1+\sigma}\cdot\xi\frac{d}{d\xi} - \frac{3\sigma}{1+\sigma}k\right)\Theta_{m0}(\xi) = 0. \tag{41}$$

The solution of this equation which decays to zero when $\xi \to \infty$ is

$$\Theta_{m0} = A\exp\left(-\frac{\bar{k}\xi^2}{8}\right)\cdot U\left(\frac{7}{2}, \sqrt{\frac{\bar{k}}{2}}\xi\right), \tag{42}$$

where $U(7/2, \sqrt{\bar{k}/2}\xi)$ is a parabolic cylinder function, A is a constant and $\bar{k}$ is defined by

$$\bar{k} = \frac{2\sigma}{1+\sigma}k. \tag{43}$$

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On using (42) in (40), we obtain

$$-\frac{\bar{k}C}{2} - \sqrt{\frac{\bar{k}}{2}} \frac{4U\left(9/2, \sqrt{\bar{k}/2}C\right)}{U\left(7/2, \sqrt{\bar{k}/2}C\right)} = \frac{1 - C^2/4}{C - C^3/12},$$

$$A = \frac{C - C^3/12}{2U(7/2, \sqrt{\bar{k}/2}C)} \cdot \exp\left(\frac{\bar{k}C^2}{8}\right). \tag{44}$$

The first of these relations determines C, the location of the upper transition layer. It is easy to see that if we multiply both sides by the denominator of the right hand side and then move the numerator to the left hand side, the resulting algebraic equation changes sign at places where the denominator and the numerator vanish; so there is a single solution which lies in the interval $2 < C < 2\sqrt{3}$ for all $\bar{k}$.

The solution for the lower transition layer can be obtained in a similar fashion. It can be shown to be given by (42) with A replaced by $-A$ and ξ replaced by $-\xi$. Thus by (31), (32), (34), (37) and (42), for a given $X \gg 1$ the flow structure in the interval $-\infty < \phi < \infty$ is completely determined.

Since the amplitude of V_0 and θ_{m0} grows as the vortices propagate downstream, insertion of (31) and (32) back into (14) shows that the latter expansions become invalid when $\tilde{x} = O(\epsilon^{-1})$ where the mean flow corrections $\epsilon^{3/2}u_{m0}$ and $\epsilon^{3/2}\theta_{m0}$ become as large as the basic state. Since $\phi = \epsilon^{-1/2}(\eta - \eta^*)$, at $\tilde{x} = O(\epsilon^{-1})$ (i.e. $x - x_n = O(1)$) the transition layers are at $\eta - \eta^* = O(1)$ and are of thickness of order $\epsilon^{1/2}\tilde{x}^{-1/6} = \epsilon^{2/3}$. In the next section, we shall consider the further downstream development of these large amplitude vortices beyond $x - x_n = O(1)$.

4 The fully nonlinear theory

It has been shown in the previous section that at positions $O(1)$ downstream of the neutral position x_n, the mean temperature correction and the mean streamwise velocity become as large as the basic state. When this happens, we expect that the large M structure of the boundary layer is still valid. The total flow is now written as

$$u = \bar{u} + \frac{1}{M_1}U, \quad v = \bar{v} + V, \quad w = W,$$

$$p = \bar{p} + \frac{1}{R}(\tilde{p} + P), \quad T = \bar{T} + T. \tag{1}$$

It should be noted, however, that although (1) is of the same form as (3), $(\bar{u}, \bar{v}, \bar{p}, \bar{T})$ here is the non-harmonic part of the total flow and is different from its counterpart in (3) which represents the unperturbed basic state. In the temperature adjustment layer, the similarity

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variable $\eta = \eta(x, y)$ is defined by

$$\eta = \frac{1}{\sqrt{2x}} \int_0^\nu \frac{dy}{\bar{T}(x, \eta(x,y))}, \tag{2}$$

where the function $\bar{T}$ in the integrand is understood to be the composite solution of the mean temperature (i.e. the wall layer temperature plus the mean temperature in the temperature adjustment layer). We note that because of the $O(1)$ correction from nonlinear interaction, the mean temperature is now also a function of x. In the limit $\eta \to 0$ or $x - x_n \to 0$, $\bar{T}(x, \eta) \to \bar{T}(\eta)$ and (2) then reduces to (11).

We assume that the mean velocity components $\bar{u}$ and $\bar{v}$ have the following expressions:

$$\bar{u} = \frac{\partial f(x, \eta)}{\partial \eta}, \quad \bar{v} = \frac{1}{\sqrt{2x}} \left\{ -\bar{T}(x, \eta) f + \frac{\partial f}{\partial \eta} I(\bar{T}) \right\} + v_\delta(x, \eta). \tag{3}$$

Here $f(x, \eta)$ and $\bar{T}(x, \eta)$ expand as

$$f(x, \eta) = \eta - \frac{\beta}{M^{1/2}} + \frac{\bar{f}(x, \eta)}{M_1} + \cdots, \quad \bar{T} = \bar{T}(x, \eta) + \cdots. \tag{4}$$

The function $v_\delta(x, \eta)$ in (3b) is added in order to satisfy the continuity equation. As can be seen from (31), this added term is partly due to the dependence of $\bar{T}$ on x and partly due to the $O(1)$ mean flow correction from nonlinar terms in the continuity equation. The function $I(\bar{T})$ in (3b) denotes the integration of the mean temperature from 0 to η. It includes the contribution from the integration of the wall layer temperature and thus has the expression

$$I(\bar{T}) = M^{3/2} B + \int_0^\eta \left(\bar{T}(x, \xi) - \frac{9(1+m)^2}{\sigma^2 \xi^4} \right) d\xi - \frac{3(1+m)^2}{\sigma^2 \eta^3} \tag{5}$$

where the constant B is defined by (20a).

With the aid of (2)–(4), the following important relations can easily be established:

$$\eta_x = -\frac{I(\bar{T})}{2x\bar{T}} - \frac{1}{\bar{T}} \frac{\partial I(\bar{T})}{\partial x}, \tag{6}$$

$$\bar{u} \frac{\partial}{\partial x} + \bar{v} \frac{\partial}{\partial y} = \frac{\partial}{\partial x} + \left\{ \frac{v_\delta}{\sqrt{2x}\bar{T}} - \frac{\eta}{2x} - \frac{1}{\bar{T}} \frac{\partial I(\bar{T})}{\partial x} \right\} \frac{\partial}{\partial \eta}, \tag{7}$$

$$\bar{u} = 1 + \frac{1}{M_1} \frac{\partial \bar{f}}{\partial \eta} + \cdots, \quad \bar{v} = \frac{B}{\sqrt{2x}} M^{3/2} + D(x, \eta), \tag{8}$$

where $D(x, \eta)$ is defined by

$$D(x, \eta) = \frac{1}{\sqrt{2x}} \left\{ \int_0^\eta \left(\bar{T}(x, \xi) - \frac{9(1+m)^2}{\sigma^2 \xi^4} \right) d\xi - \frac{3(1+m)^2}{\sigma^2 \eta^3} \right\} + v_\delta - \frac{\eta \bar{T}}{\sqrt{2x}}. \tag{9}$$

The operator on the left hand side of (7) will frequently appear in our following analysis. To simplify notation, we shall denote it by $L()$. Thus for any function $F(x, \eta)$ we have

$$L(F) = \frac{\partial F}{\partial x} + \left\{ \frac{v_\delta}{\sqrt{2x}\bar{T}} - \frac{\eta}{2x} - \frac{1}{\bar{T}} \frac{\partial I(\bar{T})}{\partial x} \right\} \frac{\partial F}{\partial \eta}. \tag{10}$$

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By substituting (1) into the Navier-Stokes equations (4)–(9), making use of the above relations and then neglecting all the cubic and higher order nonlinear terms of (U, V, W, P, T), we obtain the following perturbation equations:

$$\frac{1}{\bar{T}}L(\bar{f}') - \frac{1}{2x\bar{T}}\frac{\partial}{\partial\eta}(\frac{\bar{\mu}\bar{f}''}{\bar{T}}) - \frac{1}{\bar{T}}L(U) - \bar{\mu}U_{zz} - \frac{1}{2x\bar{T}}\frac{\partial}{\partial\eta}(\frac{\bar{\mu}}{\bar{T}}\frac{\partial U}{\partial\eta}) + \frac{\bar{f}''}{\sqrt{2x}\bar{T}^2}V$$

$$-[\frac{1}{\bar{T}^2}L(\bar{f}') + \frac{1}{2x\bar{T}}\frac{\partial}{\partial\eta}(\frac{\bar{\mu}}{\bar{T}}\bar{f}'')]T - \frac{\bar{\mu}\bar{f}''}{2x\bar{T}^2}\frac{\partial T}{\partial\eta} + \frac{1}{\bar{T}}(\frac{1}{\sqrt{2x\bar{T}}}V\frac{\partial U}{\partial\eta} + WU_z)$$

$$-\bar{\mu}(TU_z)_z - \frac{T}{\bar{T}^2}[L(U) + \frac{\bar{f}''}{\sqrt{2x\bar{T}}}V] - \frac{1}{2x\bar{T}}\frac{\partial}{\partial\eta}(\frac{\bar{\mu}}{\bar{T}}T\frac{\partial U}{\partial\eta})$$

$$+ L(\bar{f}')\frac{T^2}{\bar{T}^3} - \frac{1}{4x\bar{T}}\frac{\partial}{\partial\eta}(\frac{\bar{\mu}\bar{f}''}{\bar{T}}T^2) = 0, \tag{11}$$

$$\frac{1}{\bar{T}}L(\bar{v}) + \frac{1}{2\bar{T}}\cdot G\kappa\bar{u}^2 + \frac{1}{\sqrt{2x\bar{T}}}\frac{\partial\tilde{p}}{\partial\eta} - \frac{4}{6x\bar{T}}\frac{\partial}{\partial\eta}(\frac{\bar{\mu}\bar{v}'}{\bar{T}})$$

$$+\frac{1}{\bar{T}}L(V) - \frac{\eta\bar{T}'}{2x\bar{T}^2}V - \bar{\mu}V_{zz} - \frac{4}{3\cdot(2x)\bar{T}}\frac{\partial}{\partial\eta}(\frac{\bar{\mu}}{\bar{T}}\frac{\partial V}{\partial\eta}) + \frac{1}{\sqrt{2x\bar{T}}}\frac{\partial P}{\partial\eta}$$

$$-\left\{\frac{1}{2}G\kappa\bar{u}^2 - \frac{B}{(2x)^{3/2}}M^{3/2} + L(D) + \frac{2\bar{T}}{3x}\frac{\partial}{\partial\eta}(\frac{\bar{\mu}\bar{v}'}{\bar{T}})\right\}\frac{T}{\bar{T}^2} - \frac{2\bar{\mu}\bar{v}'}{3x\bar{T}^2}\frac{\partial T}{\partial\eta}$$

$$+\frac{2\bar{\mu}\bar{T}'}{3\sqrt{2x\bar{T}}}W_z - \frac{\bar{\mu}}{3\sqrt{2x\bar{T}}}\frac{\partial}{\partial\eta}(W_z) + \frac{1}{\bar{T}}(\frac{1}{\sqrt{2x\bar{T}}}V\frac{\partial V}{\partial\eta} + WV_z)$$

$$-\frac{T}{\bar{T}^2}(L(V) + \frac{\bar{v}'}{\sqrt{2x\bar{T}}}V) + [\frac{2}{3}\bar{\mu}W_z - \frac{4\bar{\mu}}{3\sqrt{2x\bar{T}}}\frac{\partial V}{\partial\eta}]\cdot\frac{1}{\sqrt{2x\bar{T}}}\frac{\partial T}{\partial\eta}$$

$$-\left\{\frac{4}{3\cdot(2x)\bar{T}^2}\frac{\partial\bar{\mu}}{\partial\eta}\frac{\partial V}{\partial\eta} + \frac{4\bar{\mu}}{3\cdot(2x)\bar{T}}\frac{\partial}{\partial\eta}(\frac{1}{\bar{T}}\frac{\partial V}{\partial\eta}) + \frac{\bar{\mu}}{3\sqrt{2x\bar{T}}}\frac{\partial W_z}{\partial\eta}\right.$$

$$\left.-\frac{2}{3\sqrt{2x\bar{T}}}\frac{\partial\bar{\mu}}{\partial\eta}W_z + \bar{\mu}V_{zz}\right\}T - [\frac{1}{\sqrt{2x\bar{T}}}\frac{\partial W}{\partial\eta} + V_z]\bar{\mu}T_z$$

$$+\frac{T^2}{\bar{T}^3}[L(\bar{v}) + \frac{1}{2}G\kappa\bar{u}^2] - \frac{1}{6x\bar{T}}\frac{\partial}{\partial\eta}\left(\frac{\bar{\mu}\bar{v}'}{2\bar{T}}T^2\right) = 0, \tag{12}$$

$$\frac{\bar{\mu}\bar{T}'}{\sqrt{2x\bar{T}}}V_z + \frac{\bar{\mu}}{3\sqrt{2x\bar{T}}}\frac{\partial V_z}{\partial\eta} - P_z + \frac{2\bar{\mu}\eta\bar{T}'}{3\cdot(2x)\bar{T}}T_z - \frac{1}{\bar{T}}(\frac{\partial W}{\partial x} - \frac{\eta}{2x}\frac{\partial W}{\partial\eta})$$

$$+\frac{4}{3}\bar{\mu}W_{zz} + \frac{1}{2x\bar{T}}\frac{\partial}{\partial\eta}(\frac{\bar{\mu}}{\bar{T}}\frac{\partial W}{\partial\eta}) - \frac{1}{\bar{T}}(\frac{1}{\sqrt{2x\bar{T}}}V\frac{\partial W}{\partial\eta} + WW_z)$$

$$+\frac{T}{\bar{T}^2}(\frac{\partial W}{\partial x} - \frac{\eta}{2x}\frac{\partial W}{\partial\eta}) + \left\{\frac{4}{3}\bar{\mu}W_{zz} + \frac{\bar{\mu}}{3\sqrt{2x\bar{T}}}\frac{\partial V_z}{\partial\eta} + \frac{1}{\sqrt{2x\bar{T}}}\frac{\partial\bar{\mu}}{\partial\eta}V_z\right.$$

$$\left.+\frac{1}{2x\bar{T}^2}\frac{\partial\bar{\mu}}{\partial\eta}\frac{\partial W}{\partial\eta} + \frac{\bar{\mu}}{2x\bar{T}}\frac{\partial}{\partial\eta}(\frac{1}{\bar{T}}\frac{\partial W}{\partial\eta})\right\}T + \frac{\bar{\mu}}{\sqrt{2x\bar{T}}}(\frac{1}{\sqrt{2x\bar{T}}}\frac{\partial W}{\partial\eta} + V_z)\frac{\partial T}{\partial\eta}$$

$$+(\frac{4}{3}W_z - \frac{2}{3\sqrt{2x\bar{T}}}\frac{\partial V}{\partial\eta})\bar{\mu}T_z = 0, \tag{13}$$

$$-\frac{1}{\bar{T}^2}L(\bar{T}) + \frac{\bar{v}'}{\sqrt{2x\bar{T}^2}} + \frac{1}{\sqrt{2x\bar{T}}}\frac{\partial}{\partial\eta}(\frac{V}{\bar{T}}) + \frac{1}{\bar{T}}W_z - \frac{\bar{v}'}{\sqrt{2x\bar{T}^3}}\cdot T - L(\frac{T}{\bar{T}^2})$$

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$$-(\frac{1}{\sqrt{2x\bar{T}}}\frac{\partial V}{\partial \eta} + W_z)\frac{T}{\bar{T}^2} - (\frac{1}{\sqrt{2x\bar{T}}}V\frac{\partial}{\partial \eta} + W\frac{\partial}{\partial z})(\frac{T}{\bar{T}^2})$$

$$+ L(\frac{T^2}{\bar{T}^3}) + \frac{\bar{v}'}{\sqrt{2x\bar{T}^4}}T^2 = 0, \tag{14}$$

$$\frac{1}{\bar{T}}L(\bar{T}) - \frac{1}{2x\sigma\bar{T}}\frac{\partial}{\partial \eta}\left(\frac{\bar{\mu}\bar{T}'}{\bar{T}}\right) + \frac{\bar{T}'}{\sqrt{2x}\bar{T}^2}V + \frac{1}{\bar{T}}L(T) - [\frac{L(\bar{T})}{\bar{T}^2} + \frac{1}{2x\sigma\bar{T}}\frac{\partial}{\partial \eta}(\frac{\bar{\mu}\bar{T}'}{\bar{T}})]T$$

$$-\frac{\bar{\mu}}{\sigma}T_{zz} - \frac{\bar{\mu}\bar{T}'}{2x\bar{T}^2\sigma}\frac{\partial T}{\partial \eta} - \frac{1}{2x\sigma\bar{T}}\frac{\partial}{\partial \eta}(\frac{\bar{\mu}}{\bar{T}}\frac{\partial T}{\partial \eta}) + \frac{1}{\bar{T}}(\frac{1}{\sqrt{2x\bar{T}}}V\frac{\partial T}{\partial \eta} + WT_z)$$

$$-\frac{T}{\bar{T}^2}(L(T) + \frac{\bar{T}'}{\sqrt{2x\bar{T}}}V) - \frac{1}{2x\sigma\bar{T}^2}\frac{\partial \bar{\mu}}{\partial \eta}T\frac{\partial T}{\partial \eta}$$

$$-\frac{\bar{\mu}}{\sigma}[\frac{1}{2x\bar{T}}\frac{\partial}{\partial \eta}(\frac{T}{\bar{T}}\frac{\partial T}{\partial \eta}) + (TT_z)_z] + \frac{L(\bar{T})}{\bar{T}^3}T^2 - \frac{1}{4\sigma x\bar{T}}\frac{\partial}{\partial \eta}\left(\frac{\bar{\mu}\bar{T}'}{\bar{T}}T^2\right) = 0. \tag{15}$$

Here, to simplify notation, we have used a prime to denote partial differentiation of the mean flow quantities with respect to η. As $x - x_n \to 0$, the mean corrections produced by nonlinear interaction become increasingly small and the above equations then reduce to two sets of equations: the basic state equations (13) and (14) and the perturbation equations (4)–(8).

In the light of the results given in the previous section, we expect that Görtler vortices would be trapped in an $O(1)$ region bounded by two transition layers centred at $\eta = \eta_1(x)$ and $\eta_2(x)$, each of which has thickness of order $O(\epsilon^{3/2})$. The configuration is sketched in Figure 1 in which the region of vortex activity is denoted by I, the upper and the lower transition layers by IIa and IIa respectively, whilst the region above the upper transition layer and the region below the lower transition layer are denoted by IIIa and IIIb, respectively. The flow properties in these regions are now considered separately.

We start with the core region I. There the sizes of the perturbation quantities can be determined from the results given in the previous section. From (25) and (31) we deduce that at $x - x_n = O(1)$,

$$V_0 = O(\epsilon^{-1/2}), \quad \frac{\partial V_0}{\partial \phi} = O(1).$$

Relations (17) then give

$$U_0 = O(\epsilon^{-1/2}), \quad \theta_0 = O(\epsilon^{-1/2}), \quad W_0 = O(1), \quad P_0 = O(1). \tag{16}$$

Then from the relations between $(U_0, V_0, W_0, \theta_0, P_0)$ and (U, V, W, θ, P) shown in (14), we deduce that

$$U = O(\epsilon), \quad V = O(\epsilon^{-1}), \quad W = (1), \quad T = O(\epsilon), \quad P = O(\epsilon^{-1}). \tag{17}$$

We therefore assume the following form of solutions for (11)–(15):

$$\frac{\partial \bar{f}}{\partial \eta} = \bar{f}_0(x, \eta) + \epsilon \bar{f}_1(x, \eta) + \cdots,$$

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$$\bar{T}(x,\eta) = \bar{T}_0(x,\eta) + \epsilon \bar{T}_1(x,\eta) + \cdots,$$

$$v_\delta = v_\delta^0(x,\eta) + \epsilon v_\delta^1(x,\eta) + \cdots,$$

$$U = \epsilon \left\{ E(U_0^1 + \epsilon U_1^1 + \cdots) + \epsilon E^2(U_0^2 + \cdots) + \cdots + C.C. \right\},$$

$$V = \epsilon^{-1} \left\{ E(V_0^1 + \epsilon V_1^1 + \cdots) + \epsilon E^2(V_0^2 + \cdots) + \cdots + C.C. \right\}, \qquad (18)$$

$$W = \left\{ E(W_0^1 + \epsilon W_1^1 + \cdots) + \epsilon E^2(W_0^2 + \cdots) + \cdots + C.C. \right\},$$

$$P = \epsilon^{-1} \left\{ E(P_0^1 + \epsilon P_1^1 + \cdots) + \epsilon E^2(P_0^2 + \cdots) + \cdots + C.C. \right\},$$

$$T = \epsilon \left\{ E(\theta_0^1 + \epsilon \theta_1^1 + \cdots) + \epsilon E^2(\theta_0^2 + \cdots) + \cdots + C.C. \right\},$$

where E is defined as in (15) and C.C. denotes the conjugate. We now substitute these expansions into the perturbation equations (11)–(15). After equating the coefficients of $E^0 \epsilon^0$ in (14), (11) and (15), we obtain

$$\frac{\partial \bar{T}_0}{\partial x} + \frac{v_\delta^0}{\sqrt{2x}\bar{T}_0}\frac{\partial \bar{T}_0}{\partial \eta} - \frac{1}{\bar{T}_0}\frac{\partial \bar{T}_0}{\partial \eta} \cdot \frac{\partial I(\bar{T}_0)}{\partial x} - \frac{1}{\sqrt{2x}}\frac{\partial v_\delta^0}{\partial \eta}$$

$$= -\frac{2\bar{T}_0}{\sqrt{2x}H(x)}\frac{\partial}{\partial \eta}(\bar{\mu}_0|V_0^1|^2), \qquad (19)$$

$$\frac{\partial}{\partial \eta}\left(\frac{\bar{\mu}_0}{\bar{T}_0}\frac{\partial \bar{f}_0}{\partial \eta}\right) + \eta\frac{\partial \bar{f}_0}{\partial \eta} - 2x\frac{\partial \bar{f}_0}{\partial x} + \left(2x\frac{\partial I(\bar{T}_0)}{\partial x} - \sqrt{2x}v_\delta^0\right)\frac{1}{\bar{T}_0}\frac{\partial \bar{f}_0}{\partial \eta}$$

$$= \frac{\sqrt{2x}}{\bar{T}_0}V_0^1\frac{\partial \bar{U}_0^1}{\partial \eta} + 2xU_0^1 \cdot i\bar{W}_0^1 - \frac{\sqrt{2x}}{\bar{T}_0^2}\frac{\partial \bar{f}_0}{\partial \eta}\theta_0^1\bar{V}_0^1 + C.C., \qquad (20)$$

$$\frac{1}{\sigma}\frac{\partial}{\partial \eta}\left(\frac{\bar{\mu}_0}{\bar{T}_0}\frac{\partial \bar{T}_0}{\partial \eta}\right) + \eta\frac{\partial \bar{T}_0}{\partial \eta} - 2x\frac{\partial \bar{T}_0}{\partial x} + \left(2x\frac{\partial I(\bar{T}_0)}{\partial x} - \sqrt{2x}v_\delta^0\right)\frac{1}{\bar{T}_0}\frac{\partial \bar{T}_0}{\partial \eta}$$

$$= \frac{\sqrt{2x}}{\bar{T}_0}V_0^1\frac{\partial \bar{\theta}_0^1}{\partial \eta} + 2x\theta_0^1 \cdot i\bar{W}_0^1 - \frac{\sqrt{2x}}{\bar{T}_0^2}\frac{\partial \bar{T}_0}{\partial \eta}\theta_0^1\bar{V}_0^1 + C.C., \qquad (21)$$

where $\bar{\mu}_0 = \bar{\mu}(\bar{T}_0)$, and a bar over U and W signifies conjugation.

On equating the coefficients of $E\epsilon^{-1}$ in (11), $E\epsilon^{-3}$ in (12), $E\epsilon^{-2}$ in (13), $E\epsilon^{-1}$ in (14) and $E\epsilon^{-1}$ in (15), we have

$$\bar{\mu}_0 U_0^1 + \frac{1}{\sqrt{2x}\bar{T}_0^2}\frac{\partial \bar{f}_0}{\partial \eta}V_0^1 = 0, \qquad (22)$$

$$\bar{\mu}_0 V_0^1 - \frac{H(x)}{\bar{T}_0^2}\theta_0^1 = 0, \qquad (23)$$

$$\frac{\bar{\mu}_0}{\sqrt{2x}\bar{T}_0}\frac{\partial \bar{T}_0}{\partial \eta}V_0^1 + \frac{\bar{\mu}_0}{3\sqrt{2x}\bar{T}_0}\frac{\partial V_0^1}{\partial \eta} - P_0^1 + \frac{4}{3}\bar{\mu}_0(iW_0^1) = 0, \qquad (24)$$

$$\frac{\partial}{\partial \eta}\left(\frac{V_0^1}{\bar{T}_0}\right) + \sqrt{2x}(iW_0^1) = 0, \qquad (25)$$

$$\frac{1}{\sqrt{2x}\bar{T}_0^2}\frac{\partial \bar{T}_0}{\partial \eta}V_0^1 + \frac{1}{\sigma}\bar{\mu}_0\theta_0^1 = 0, \qquad (26)$$

where $\bar{\mu}_0 = \bar{\mu}(\bar{T}_0)$ and where $H(x)$ is defined as

$$H(x) = BN \left\{ \frac{\kappa(x)}{\kappa(x_n)(2x_n)^{3/2}} - \frac{1}{(2x)^{3/2}} \right\} + \frac{1}{2}\kappa g_0. \tag{27}$$

Equations (23) and (26) are consistent only if

$$\frac{\sigma}{\bar{\mu}_0^2 \bar{T}_0^4} \frac{\partial \bar{T}_0}{\partial \eta} + \frac{\sqrt{2x}}{H(x)} = 0. \tag{28}$$

This is a first order differential equation which can be solved to give an expression for the mean temperature. Note that in the limit $x \to x_n$, (28) reduces to (16) which is the condition for the original basic state to be neutrally stable at a single point $\eta = \eta^*$. Equation (28) shows that at $O(1)$ distance downstream of the neutral position, the basic state is forced by the vortices to be such that it is neutrally stable everywhere simultaneously in the region of vortex activity.

With the aid of the relations (22)–(26), we can express U_0^1, W_0^1, θ_0^1 and P_0^1 in terms of V_0^1 as follows:

$$\theta_0^1 = -\frac{\sigma}{\sqrt{2x}\bar{\mu}_0 \bar{T}_0^2} \frac{\partial \bar{T}_0}{\partial \eta} V_0^1, \quad U_0^1 = -\frac{1}{\sqrt{2x}\bar{\mu}_0 \bar{T}_0^2} \frac{\partial \bar{f}_0}{\partial \eta} V_0^1,$$

$$iW_0^1 = -\frac{1}{\sqrt{2x}} \frac{\partial}{\partial \eta} \left(\frac{V_0^1}{\bar{T}_0} \right), \tag{29}$$

$$P_0^1 = -\frac{\bar{\mu}_0}{\sqrt{2x}\bar{T}_0} \frac{\partial V_0^1}{\partial \eta} + \left(\bar{\mu}_0 + \frac{4\bar{\mu}_0}{3\bar{T}_0} \right) \frac{1}{\sqrt{2x}\bar{T}_0} \frac{\partial \bar{T}_0}{\partial \eta} V_0^1.$$

With the use of these relations, equation (19) reduces to

$$\frac{\partial}{\partial \eta} \left(\frac{v_\delta^0}{\bar{T}_0} \right) = \sqrt{2x} \frac{\partial}{\partial \eta} \left(\frac{1}{\bar{T}_0} \frac{\partial I(\bar{T}_0)}{\partial x} \right) + \frac{2}{H(x)} \frac{\partial}{\partial \eta} \left(\bar{\mu}_0 |V_0^1|^2 \right). \tag{30}$$

Integrating this equation then gives

$$v_\delta^0 = \sqrt{2x} \frac{\partial I(\bar{T}_0)}{\partial x} + \frac{2\bar{\mu}_0 \bar{T}_0}{H(x)} |V_0^1|^2. \tag{31}$$

Here we have put the arbitrary integration function of x to zero. Even if we had not done so, this function could be shown to be zero at a later stage.

With the aid of the relations (29)–(31), we can reduce (20) and (21) to

$$\frac{\partial}{\partial \eta} \left(\frac{\bar{\mu}_0}{\bar{T}_0} \frac{\partial \bar{f}_0}{\partial \eta} \right) + \eta \frac{\partial \bar{f}_0}{\partial \eta} - 2x \frac{\partial \bar{f}_0}{\partial x} = -\frac{2}{\bar{T}_0^2} \frac{\partial}{\partial \eta} \left(\frac{1}{\bar{\mu}_0 \bar{T}_0} \frac{\partial \bar{f}_0}{\partial \eta} |V_0^1|^2 \right) - \frac{4\sqrt{2x}}{\sigma H(x)} \bar{\mu}_0 \frac{\partial \bar{f}_0}{\partial \eta} |V_0^1|^2, \tag{32}$$

$$\frac{1}{\sigma} \frac{\partial}{\partial \eta} \left(\frac{\bar{\mu}_0}{\bar{T}_0} \frac{\partial \bar{T}_0}{\partial \eta} \right) + \eta \frac{\partial \bar{T}_0}{\partial \eta} - 2x \frac{\partial \bar{T}_0}{\partial x} = \frac{2\sqrt{2x}\bar{T}_0}{H(x)} \cdot \frac{\partial}{\partial \eta} (\bar{\mu}_0 \bar{T}_0 |V_0^1|^2). \tag{33}$$

After $\bar{T}_0(x, \eta)$ has been determined from (28), equation (33) can be used to determine $|V_0^1|$ and solving (32) then gives an expression for $\bar{f}_0(x, \eta)$. From (28) we see that the boundary layer flow is forced by the vortices which, from (32), are driven by the boundary layer.

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We now proceed to solve (28) to determine the mean temperature $\bar{T}_0(x, \eta)$. When $\bar{\mu}$ is given by Sutherland's law (16), (28) becomes

$$\frac{(\bar{T}_0 + m)^2}{\bar{T}_0^7} \frac{\partial \bar{T}_0}{\partial \eta} = -\frac{(1 + m)^2}{\sigma} \cdot \frac{\sqrt{2x}}{H(x)}. \tag{34}$$

Integrating (34) gives

$$\frac{1}{\bar{T}_0^6}\left(\frac{1}{6}m^2 + \frac{2}{5}m\bar{T}_0 + \frac{1}{4}\bar{T}_0^2\right) = \frac{(1 + m)^2}{\sigma} \cdot \frac{\sqrt{2x}}{H(x)}(\eta + a(x)), \tag{35}$$

where $a(x)$ is a function to be determined.

Integrating (33) with the aid of (34) yields

$$|V_0^1|^2 = \frac{H(x)}{2\sqrt{2x}\bar{\mu}_0\bar{T}_0} \int^\eta \frac{1}{\bar{T}_0}\left\{\frac{1}{\sigma}\frac{\partial}{\partial\eta}\left(\frac{\bar{\mu}_0}{\bar{T}_0}\frac{\partial\bar{T}_0}{\partial\eta}\right) + \eta\frac{\partial\bar{T}_0}{\partial\eta} - 2x\frac{\partial\bar{T}_0}{\partial x}\right\} d\eta. \tag{36}$$

We now assume that $|V_0^1|^2$ vanishes at $\eta = \eta_1(x), \eta_2(x)$, which bound the region of vortex activity. As in the case of weakly nonlinear development discussed in the previous section, these two boundaries are also where the solution (36) breaks down and where transition layers exist. The thickness of each of these layers is $O(\epsilon^{2/3})$ so that in the upper transition layer at $\eta = \eta_2(x)$, we define

$$\xi = \frac{\eta - \eta_2}{\epsilon^{2/3}}. \tag{37}$$

Near $\eta = \eta_2$, we can deduce from (36) and (29) that V_0^1, θ_0^1 and U_0^1 are all of order $\epsilon^{1/3}$, and that W_0^1 and P_0^1 are both of order $\epsilon^{-1/3}$. With the use of these results, the sizes of (U, V, W, T, P) can be determined from (18), which are shown in the following asymptotic expansions:

$$U = \epsilon^{4/3}\left\{E(U_{01} + \epsilon^{2/3}U_{11} + \cdots) + \cdots + C.C.\right\},$$

$$V = \epsilon^{-2/3}\left\{E(V_{01} + \epsilon^{2/3}V_{11} + \cdots) + \cdots + C.C.\right\},$$

$$W = \epsilon^{-1/3}\left\{E(W_{01} + \epsilon^{2/3}W_{11} + \cdots) + \cdots + C.C.\right\},$$

$$P = \epsilon^{-4/3}\left\{E(P_{01} + \epsilon^{2/3}P_{11} + \cdots) + \cdots + C.C.\right\}, \tag{38}$$

$$T = \epsilon^{4/3}\left\{E(\theta_{01} + \epsilon^{2/3}\theta_{11} + \cdots) + \cdots + C.C.\right\},$$

$$\frac{\partial \bar{f}}{\partial \eta} = \hat{f}_0(x, \xi) + \epsilon^{2/3}\hat{f}_1(x, \xi) + \epsilon^{4/3}\hat{f}_2(x, \xi) + \cdots,$$

$$\bar{T} = \hat{T}_0(x, \xi) + \epsilon^{2/3}\hat{T}_1(x, \xi) + \epsilon^{4/3}\hat{T}_2(x, \xi) + \cdots,$$

$$v_\delta = \hat{v}_{\delta 0}(x, \xi) + \epsilon^{2/3}\hat{v}_{\delta 1}(x, \xi) + \epsilon^{4/3}\hat{v}_{\delta 2}(x, \xi) + \cdots.$$

Here we expect that the first two terms in (38f, g, h) are simply the expansions of the mean flow functions $\bar{f}_0(x, \eta), \bar{T}_0(x, \eta)$ and v_δ^0 in I near $\eta = \eta_2$. Hence

$$\hat{f}_0(x, \xi) = \bar{f}_0(x, \eta_2), \quad \hat{f}_1(x, \xi) = \frac{\partial \bar{f}_0(x, \eta_2)}{\partial \eta} \cdot \xi,$$

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$$\hat{T}_0(x,\xi) = \bar{T}_0(x,\eta_2) \quad \hat{T}_1(x,\xi) = \frac{\partial \bar{T}_0(x,\eta_2)}{\partial \eta} \cdot \xi. \tag{39}$$

Similar expressions can be written down for $\hat{v}_{\delta 0}$ and $\hat{v}_{\delta 1}$. On substituting (38a-g) into (11)–(15) and equating the coefficients of various orders of the harmonics and mean flow quantities, we obtain a hierarchy of equations. From (12) we have from equating the coefficients of $E\epsilon^{-8/3}$ and $E\epsilon^{-2}$,

$$\bar{\mu}_0 V_{01} - \frac{H(x)}{\bar{T}_0^2}\theta_{01} = 0, \tag{40}$$

$$\bar{\mu}_0 V_{11} + \bar{\mu}_0 \xi \bar{T}_0' V_{01} - \frac{H(x)}{\bar{T}_0^2}(\theta_{11} - \frac{2\bar{T}_0'\xi}{\bar{T}_0}\theta_{01})$$

$$- \frac{4\bar{\mu}_0}{3(2x)\bar{T}_0^2}\frac{\partial^2 V_{01}}{\partial \xi^2} + \frac{1}{\sqrt{2x}\bar{T}_0}\frac{\partial P_{01}}{\partial \xi} - \frac{\bar{\mu}_0}{3\sqrt{2x}\bar{T}_0}\frac{\partial(iW_{01})}{\partial \xi} = 0. \tag{41}$$

Here and subsequently we write $\bar{T}_0'$ for $\partial \bar{T}_0(x,\eta_2)/\partial \eta$ to simplify the notation and $\bar{\mu}_0 = \bar{\mu}(\bar{T}_0(x,\eta_2)), \dot{\bar{\mu}}_0 = \dot{\bar{\mu}}(\bar{T}_0(x,\eta_2))$. Equating the coefficients of $E\epsilon^{-7/3}$ and of $E\epsilon^{-4/3}$ in (14) gives

$$\frac{\bar{\mu}_0}{3\sqrt{2x}\bar{T}_0}\frac{\partial V_{01}}{\partial \xi} - P_{01} + \frac{4}{3}\bar{\mu}_0(iW_{01}) = 0, \tag{42}$$

$$\frac{1}{\sqrt{2x}\bar{T}_0^2}\frac{\partial V_{01}}{\partial \xi} + \frac{iW_{01}}{\bar{T}_0} = 0. \tag{43}$$

Finally, by equating the coefficients of $E\epsilon^{-2/3}, E\epsilon^0$ and $E^0\epsilon^0$ in (15), we obtain

$$\frac{\bar{T}_0'}{\sqrt{2x}\bar{T}_0^2}V_{01} + \frac{\bar{\mu}_0}{\sigma}\theta_{01} = 0, \tag{44}$$

$$\frac{\bar{T}_0'}{\sqrt{2x}\bar{T}_0^2}V_{11} + \frac{1}{\sigma}\bar{\mu}_0\theta_{11} + \frac{1}{\sqrt{2x}\bar{T}_0^2}(\frac{\partial \hat{T}_2}{\partial \xi} - \frac{2\bar{T}_0'^2}{\bar{T}_0})V_{01}$$

$$+ \frac{\bar{\mu}_0}{\sigma}\xi \bar{T}_0'\theta_{01} - \frac{\bar{\mu}_0}{2x\sigma\bar{T}_0^2}\frac{\partial^2 \theta_{01}}{\partial \xi^2} = 0, \tag{45}$$

$$- \frac{\eta_2\bar{T}_0'}{2x\bar{T}_0} - \frac{\bar{\mu}_0}{2x\sigma\bar{T}_0^2}\frac{\partial^2 \hat{T}_2}{\partial \xi^2} - \frac{1}{2x\sigma\bar{T}_0^2}(\dot{\bar{\mu}}_0 - \frac{\bar{\mu}_0}{\bar{T}_0})\bar{T}_0'^2$$

$$+ \frac{1}{\sqrt{2x}\bar{T}_0^2}(V_{01}\frac{\partial \bar{\theta}_{01}}{\partial \xi} + \bar{V}_{01}\frac{\partial \theta_{01}}{\partial \xi}) + \frac{1}{\bar{T}_0}(i\bar{W}_{01}\theta_{01} - iW_{01}\bar{\theta}_{01}) = 0. \tag{46}$$

We have not written down the equations obtained from the x-momentum equation since the determination of $V_{01}, W_{01}, \theta_{01}$ and P_{01} does not involve the streamwise velocity component. Equations (40)–(46) can be solved in the following way. First, from (42), (43) and (44), we can express iW_{01}, P_{01} and θ_{01} in terms of V_{01} and its derivatives. We note that equations (40) and (44) are already consistent because of (28). Next, equations (41) and (45) give a 2 × 2 inhomogeneous matrix equation of the form $A\zeta = f$ for $\zeta = (V_{11}, \theta_{11})^T$. The inhomogeneous

term f involves $\partial \hat{T}_2/\partial \xi$ as well as V_{01} and its derivatives. The former can be determined from (46) and can be shown to be given by

$$\frac{\partial \hat{T}_2}{\partial \xi} = -\frac{\sigma \bar{T}_0 \eta_2 \bar{T}_0'}{\bar{\mu}_0}\xi - (\frac{\tilde{\mu}_0}{\bar{\mu}_0} - \frac{1}{\bar{T}_0})\bar{T}_0'^2\xi - \frac{2\sigma^2 \bar{T}_0'}{\bar{\mu}_0^2 \bar{T}_0^2}|V_{01}|^2 - S(x), \qquad (47)$$

where $S(x)$ is a function to be determined. Since the coefficient matrix $\mathbf{A}$ has zero determinant because (40) and (44) must have non-trivial solutions, the inner product of the left eigenvector of $\mathbf{A}$ and $\mathbf{f}$ must vanish. Omitting all the details, we can show that the latter condition can be reduced to the following second order partial differential equation for V_{01}:

$$\frac{\partial^2 V_{01}}{\partial \xi^2} + S_1(x)\xi V_{01} = \frac{4x}{3} \cdot \frac{\sigma^2}{\bar{\mu}_0^2}V_{01}|V_{01}|^2 + S_2(x)V_{01}, \qquad (48)$$

where

$$S_1(x) = -\frac{2x}{3}\bar{T}_0 \left(3\bar{T}_0' + \frac{3\tilde{\mu}_0}{\bar{\mu}_0}\bar{T}_0\bar{T}_0' + \frac{\sigma}{\bar{\mu}_0}\eta_2\bar{T}_0^2\right), \qquad (49)$$

$$S_2(x) = \frac{(2x)}{3} \cdot \frac{\bar{T}_0^2}{\bar{T}_0'} \cdot S(x). \qquad (50)$$

This equation is a particular form of the second Painleve transcendent and has been shown by Hastings and Mcleod (1978) to have a solution such that

$$\frac{4x\sigma^2}{3\bar{\mu}_0^2}|V_{01}|^2 \sim S_1(x)\xi \quad \text{as} \quad \xi \to -\infty \qquad (51)$$

and that $|V_{01}|$ decays to zero exponentially as $\xi \to \infty$. Thus the Görtler vortices are trapped below region IIIa and the condition (51) ensures that V_{01} matches with V_0^1 in the core region I. An identical analysis applied to the lower transition layer IIb shows that the vortices there are also reduced to zero exponentially away from the core region so that they are also trapped above region IIIb. As a consequence, above the upper transition layer and below the lower transition layer, there are only mean flow fields.

In IIIa,b the mean flow fields are still formally represented by (3) and (4), but now $\bar{f}$ and $\bar{T}$ expand as

$$\bar{f}(x,\eta) = \hat{f}_0(x,\eta) + O(\epsilon), \quad \bar{T}(x,\eta) = \hat{T}_0(x,\eta) + O(\epsilon), \qquad (52)$$

where $\hat{f}_0(x,\eta)$ and $\hat{T}_0(x,\eta)$ satisfy

$$(1+m)\frac{\partial}{\partial \eta}\left(\frac{\sqrt{\hat{T}_0}}{\hat{T}_0 + m}\frac{\partial^2 \hat{f}_0}{\partial \eta^2}\right) + \eta\frac{\partial^2 \hat{f}_0}{\partial \eta^2} - 2x\frac{\partial^2 \hat{f}_0}{\partial x \partial \eta} = 0,$$

$$\frac{(1+m)}{\sigma}\frac{\partial}{\partial \eta}\left(\frac{\sqrt{\hat{T}_0}}{\hat{T}_0 + m}\frac{\partial \hat{T}_0}{\partial \eta}\right) + \eta\frac{\partial \hat{T}_0}{\partial \eta} - 2x\frac{\partial \hat{T}_0}{\partial x} = 0. \qquad (53)$$

To satisfy the continuity equation, v_δ must be calculated from

$$v_\delta = \sqrt{2x}\frac{\partial I(\bar{T})}{\partial x}, \qquad (54)$$

where the function I is defined by (5). We note that the governing equation (53b) for $\hat{T}_0(x,\eta)$ is decoupled from that for $\hat{f}_0$. We can therefore solve (53b) on $[0,\eta_1]$, $[\eta_2,\infty]$ first subject to the following boundary and matching conditions:

(i).

$$\hat{T}_0(x,\eta) \rightarrow \frac{9}{\sigma^2}(1+m)^2\frac{1}{\eta^4} + \cdots, \quad \text{as} \;\; \eta \rightarrow 0, \quad \hat{T}_0(x,\eta) \rightarrow 1, \quad \text{as} \;\; \eta \rightarrow \infty; \qquad (55)$$

(ii).

$$\hat{T}_0(x,\eta_j) = \bar{T}_0(x,\;\eta_j), \quad \partial\hat{T}_0(x,\eta_j)/\partial\eta = \partial\bar{T}_0(x\;\eta_j)/\partial\eta, \quad j=1,2; \qquad (56)$$

(iii). $\quad \eta = \eta_1$ and η_2 are where V_0^1 vanishes and from (36) they satisfy

$$\int_{\eta_1}^{\eta_2} \frac{1}{\bar{T}_0} \left\{ \frac{1}{\sigma}\frac{\partial}{\partial\eta}(\frac{\bar{\mu}_0}{\bar{T}_0}\frac{\partial\bar{T}_0}{\partial\eta}) + \eta\frac{\partial\bar{T}_0}{\partial\eta} - 2x\frac{\partial\bar{T}_0}{\partial x} \right\} d\eta = 0. \qquad (57)$$

Thus equations (52b)–(57) constitute a free boundary problem which can be solved numerically for a given curvature distribution to determine the two boundaries $\eta_1(x), \eta_2(x)$ and the unknown function $a(x)$ in (35). Once these three functions are determined, all of the perturbation quantities can be calculated by the appropriate formulae given in this section.

5   Numerical results

In this section, we shall outline a numerical scheme which we have used to integrate the above free boundary problem and discuss our numerical results.

For convenience, we shall drop the hat notation and subscripts 'o' in (53)–(57). Thus our free boundary problem is to integrate

$$\frac{(1+m)}{\sigma}\frac{\partial}{\partial\eta}\left(\frac{\sqrt{T}}{T+m}\frac{\partial T}{\partial\eta}\right) + \eta\frac{\partial T}{\partial\eta} - 2x\frac{\partial T}{\partial x} = 0, \qquad (58)$$

subject to the boundary and matching conditions (55)–(57). The interval $(0,\infty)$ is divided into three sub-intervals:

$$\Gamma_1 : (0,\eta_1), \quad \Gamma_2 : (\eta_1,\eta_2), \quad \Gamma_3 : (\eta_2,\infty).$$

Our aim now is to integrate (58) in the intervals Γ_1 and Γ_3, and iterate on the values of $\eta_1(x), \eta_2(x)$ and $a(x)$ at a given x so that the matching conditions (56) at η_1 and η_2 are satisfied and the integral (57) over Γ_2 vanishes.

For the purpose of numerical calculation, it is necessary to work with fixed boundaries so in Γ_1 and Γ_3 we make the transformations

$$\eta = \eta_1(x)e^\phi, \qquad \eta = \eta_2(x)\psi, \qquad (59)$$

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respectively, so that the intervals Γ_1 and Γ_3 now become

$$\Gamma_1{}^\phi : (-\infty, 0), \quad \Gamma_3{}^\psi : (1, \infty). \tag{60}$$

The additional exponential stretching in (59a) is introduced to accomodate the rapid change of T near $\eta = 0$ (as indicated by (55)).

In terms of the new variables ϕ and ψ, (58) becomes

$$2x \frac{\partial T}{\partial x} + A_1(x, \phi, T) \frac{\partial^2 T}{\partial \phi^2} = B_1(x, \phi, T, \frac{\partial T}{\partial \phi}), \tag{61}$$

and

$$2x \frac{\partial T}{\partial x} + A_3(x, \psi, T) \frac{\partial^2 T}{\partial \psi^2} = B_3(x, \psi, T, \frac{\partial T}{\partial \psi}), \tag{62}$$

respectively, where

$$A_1 = -\frac{1+m}{\sigma} \cdot \frac{\sqrt{T}}{T+m} \cdot \frac{e^{-2\phi}}{\eta_1^2}$$

$$B_1 = \left(1 + \frac{2x\eta_1'}{\eta_1} - \frac{1+m}{\sigma} \cdot \frac{\sqrt{T}}{T+m} \cdot \frac{e^{-2\phi}}{\eta_1^2}\right) \frac{\partial T}{\partial \phi}$$

$$+ \frac{1+m}{\sigma} \cdot \frac{e^{-2\phi}}{\eta_1^2} \cdot \frac{m-T}{2\sqrt{T}(T+m)^2} \cdot (\frac{\partial T}{\partial \phi})^2, \tag{63}$$

$$A_3 = -\frac{1+m}{\sigma} \cdot \frac{\sqrt{T}}{T+m} \cdot \frac{1}{\eta_2^2}$$

$$B_3 = \left(1 + \frac{2x\eta_2'}{\eta_2}\right) \cdot \psi \frac{\partial T}{\partial \psi} + \frac{1+m}{\sigma} \cdot \frac{1}{\eta_2^2} \cdot \frac{m-T}{2\sqrt{T}(T+m)^2} \cdot (\frac{\partial T}{\partial \psi})^2.$$

Equations (61) and (62) are parabolic partial differential equations, so their solutions can be otained by a marching procedure. We shall now use the solution of (61) as an illustrative example to explain our numerical scheme. If the values of $T, \eta_1(x), \eta_2(x)$ and $a(x)$ are known at $x = \tilde{x}$, then the following schem is used to determine these functions at $\tilde{x} + \tilde{\epsilon}$:

$$2\tilde{x} \cdot \frac{\tilde{T}_i - T_i}{\tilde{\epsilon}} + A_1(\tilde{x}, \phi_i, T_i) \cdot \frac{\tilde{T}_{i+1} - 2\tilde{T}_i + \tilde{T}_{i-1}}{h^2} = B_1(\tilde{x}, \phi_i, T_i, \frac{T_{i+1} - T_{i-1}}{2h}), \tag{64}$$

where h is the vertical grid spacing, a tilde denotes a quantity evaluated at the position $\tilde{x} + \tilde{\epsilon}$ and a subscript signifies evaluation at the indicated vertical grid point. In the expression for B_1, $\eta_1'(\tilde{x})$ is replaced by

$$\eta_1'(\tilde{x}) = \frac{\tilde{\eta}_1 - \eta_1(\tilde{x})}{\tilde{\epsilon}}, \tag{65}$$

where $\tilde{\eta}_1$ is a guess for $\eta_1(\tilde{x} + \tilde{\epsilon})$. If we replace $-\infty$ by ϕ_0 and use n mesh points in the η direction, we have

$$\phi_i = \phi_0 + ih, \quad \phi_n = \phi_0 + nh = 0.$$

Application of (64) to $i = 1, 2, \cdots, n-1$ gives a triagonal matrix equation which can be solved after the following boundary conditions are incorporated:

$$T_0 = \frac{9(1+m)^2}{\sigma^2} \cdot \frac{1}{\tilde{\eta}_1^4 \cdot e^{4\phi_0}}, \quad T_n = \bar{T}_0(x, \eta_1), \tag{66}$$

where $\bar{T}_0(x, \eta_1)$ is calculated from (35). The derivative of T at $\eta = \eta_1(x)$ is calculated from

$$\frac{\partial T}{\partial \eta}\Big|_{\eta=\eta_1} = \frac{1}{\eta_1} \frac{\partial T}{\partial \phi}\Big|_{\phi=0} = \frac{T_{n+1} - T_{n-1}}{\eta_1 h}, \tag{67}$$

and we define a function f_1 by

$$f_1(\hat{a}, \tilde{\eta}_1) = \frac{\partial T}{\partial \eta}\Big|_{\eta=\eta_1} - \frac{\partial \bar{T}_0}{\partial \eta}\Big|_{\eta=\eta_1}, \tag{68}$$

where the second term is calculated from (34) and $\hat{a}$ is a guess for $a(\tilde{x} + \tilde{\epsilon})$, which is used in the calculation of $\bar{T}_0(x, \eta_1)$ according to (35). Equation (62) can be solved in a similar fashion, which leads to a second function f_2:

$$f_2(\hat{a}, \tilde{\eta}_2) \stackrel{\text{def.}}{=} \frac{1}{\eta_2} \cdot \frac{\partial T}{\partial \psi}\Big|_{\psi=1} - \frac{\partial \bar{T}_0}{\partial \eta}\Big|_{\eta=\eta_2}, \tag{69}$$

where $\tilde{\eta}_2$ is a guess for $\eta_2(\tilde{x} + \tilde{\epsilon})$. For a given guess $(\hat{a}, \tilde{\eta}_1, \tilde{\eta}_2)$, a third function f_3 is defined by means of

$$f_3(\hat{a}, \tilde{\eta}_1, \tilde{\eta}_2) = \int_{\tilde{\eta}_1}^{\tilde{\eta}_2} \frac{1}{\bar{T}_0} \left\{ \frac{1}{\sigma} \frac{\partial}{\partial \eta}\left(\frac{\bar{\mu}_0}{\bar{T}_0} \frac{\partial \bar{T}_0}{\partial \eta}\right) + \eta \frac{\partial \bar{T}_0}{\partial \eta} - 2x \frac{\partial \bar{T}_0}{\partial x} \right\} d\eta. \tag{70}$$

With the aid of the three-dimensional version of the Newton-Raphson method, our program iterate on $(\hat{a}, \tilde{\eta}_1, \tilde{\eta}_2)$ until the three error functions become sufficiently small simultaneously.

The above procedure shows how to march the values of $(a(x), \eta_1(x), \eta_2(x))$ one step forward along the stream-wise direction at a given downstream location. The scheme is complete if the initial values of $(a(x), \eta_1(x), \eta_2(x))$ are known at a certain initial position $x = x_0$. Such values are provided by the weakly nonlinear theory, as we show below.

The weakly nonlinear theory established in §4 gives the large $X(= (x - x_n)/\epsilon)$ structure for growing Görtler vortices. The present fully nonlinear theory for $x - x_n = O(1)$ should then match in the small $(x - x_n)$ limit with that large X structure. Therefore, the desired initial conditions for $(a(x), \eta_1(x), \eta_2(x))$ and T are imposed near the neutral position x_n and are obtained by rewriting the large X solutions of the weakly nonlinear theory in terms of the original variables x and η.

First of all, the initial value of $a(x)$ can be obtained by using the condition that $\tilde{\theta}_{m0}$ given by (28) vanishes at $\eta = \eta^*$ so that $\bar{T}_0$ in (35), when evaluated at $\eta = \eta^*$, can be replaced by $\bar{T}$, the unperturbed basic state temperature. To determine η_1 and η_2, we simply have to write $\xi = \pm C$ defined between (32) and (33) in terms of the original variables x and η with the aid of (9), (10), (25) and (30). The result is

$$\eta_{1,2} = \eta^* \pm C\sqrt{\frac{\tilde{b}}{4\tilde{a}}(x - x_n)}, \tag{71}$$

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where $\bar{a}$ and $\bar{b}$ are given by (23); whilst C is determined by solving (44a). Finally, the initial value of T can be written as

$$T = \bar{T} + \epsilon^{3/2}\theta_{m0}, \tag{72}$$

where $\bar{T}$ is again the unperturbed basic state temperature and the other term is the one appearing in (14d). Rewritting the latter in terms of the original variables x and η with the aid of (9), (10), (25), (39) and (42), we obtain

$$\epsilon^{3/2}\theta_{m0} = \pm A \cdot \frac{3\hat{T}_1}{2x_n\hat{T}_0^2} \cdot \frac{\bar{b}^{3/2}}{\sqrt{\bar{a}}} \cdot (x - x_n)^{3/2} \cdot \exp(-\frac{\bar{k}\xi^2}{8}) \cdot U(\frac{7}{2}, \mp\sqrt{\frac{\bar{k}}{2}}\xi), \tag{73}$$

where

$$\xi = \frac{2\sqrt{\bar{a}}}{\sqrt{\bar{b}}} \cdot \frac{\eta - \eta^*}{\sqrt{x - x_n}},$$

and where the '+' and '−' signs should be taken for $0 < \eta < \eta_1$ and $\eta_2 < \eta < \infty$ respectively.

In Fig.2, we have shown the evolution of the mean temperature correction $\epsilon^{3/2}\theta_{m0}$ given by (73) downstream of the neutral position $x_n = 0.5$; whilst in Fig.3 we have shown how the growth of $\epsilon^{3/2}\theta_{m0}$ depends upon the neutral position x_n. We can see that for a fixed value of $x - x_n$, $\epsilon^{3/2}\theta_{m0}$ decreases drastically with increasing x_n. For $x - x_n = 0.001$, our numerical calculation shows that $\epsilon^{3/2}\theta_{m0}$ becomes as small as of order 10^{-7} when $x_n = 20$. In our numerical experimentation, we find that if we choose too large a value for x_n, the amplitude of the initial Görtler vortex would be too small to have any effect on the evolution of the temperature, and as a result, the numerical values of $\eta_1(x)$ and $\eta_2(x)$ would coalesce into a single value as we march downstream. This is why we choose rather small values for x_n in our following numerical discussion. Such an experimentation also provides a check on our numerical scheme as we expect that the two free boundaries would coalesce if no vortices were present.

We now discuss our numerical results. It was found that the above numerical scheme converged for sufficiently small values of $\tilde{\epsilon}$ and that $h_1 = 0.005, h_2 = 0.004, \tilde{\epsilon} = 0.0001$ gave a stable scheme for the cases investigated and yielded values for η_1, η_2 and the other flow quantities correct to two decimal places, where h_1 and h_2 are the vertical grid spacing in Γ_1^ϕ and Γ_2^ψ, respectively. All cases correspond to $\sigma = 0.72, m = 0.509, N = 1$ and to a thermally insulated wall for which the basic state solutions have been given in Fu, Hall and Blackaby (1990). The first case we considered has the curvature distribution and the neutral position given by

$$\kappa(x) = (2x)^{3/2}, \quad x_n = 0.4. \tag{74}$$

The correponding Görtler number is from (1) given by $G = 21.8044/\epsilon^4$. The weakly nonlinear theory results (71), (72) and (73) were used to calculate the initial values of $\eta_1(x), \eta_2(x)$ and $a(x)$ and the initial profile of T at $x = 0.401$. The numerical scheme described in this

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section was then used to advance the solution beyond $x = 0.401$. The numerical values thus obtained for $\eta_1(x)$ and $\eta_2(x)$ are shown in Fig.4 and the total temperature distributions at $x = 0.4, 0.8, 1.2$ are shown in Fig.5 with that for $x = 0.4$ correponding to the unperturbed basic state temperature. In Fig.4 we have also shown the values of $\eta_1(x)$ and $\eta_2(x)$ calculated according to the weakly nonlinear theory result (71), which is strictly valid only for $x - x_n \ll 1$. In Fig.6 we have shown the eigenfunction V_0^1, calculated from (36), at the downstream locations $x = 0.5, 0.6, 0.7, 0.8$.

The second case we considered corresponds to

$$\kappa(x) = \sqrt{2x}, \quad x_n = 0.3. \tag{75}$$

The Görtler number for this case is from (1) given by $G = 18.9681/\epsilon^4$. This is the case which admits a similarity solution in the context of incompressible flows, as has been shown by Hall and Lakin (1985). Such a similarity solution is no longer possible here because of the contribution to the Görtler number expansion from the basic state curvature, as can be seen from (27) and (35). However it can be deduced from (27) that

$$H(x) \sim \kappa(x) = \sqrt{2x}, \quad \text{as} \quad x \to \infty,$$

and further from (35) that η_1 and η_2 become independent of x when x becomes large, so that a similarity solution is possible for large x. This is verified by our numercal results shown in Fig.7 which clearly shows the increasing independence of η_1 and η_2 on x when the latter becomes large.

6 Secondary instability

After the large Görtler vortex structure discussed in the previous two sections has been established, we expect that the boundary layer would become susceptible to secondary instability of the wavy vortex or vorticity mode type. Thus following Hall and Seddougui (1989), we now study the secondary instability of the steady structure described above by superimposing spanwise periodic travelling waves on the flow in the two transition layers. We shall confine our attention on the upper transition layer; the lower transition can be studied similarly. The steady vortex structure in IIa is now our basic state. We signify it by a subscript B and rewrite it here for easy reference:

$$u_B = \bar{u} + \frac{1}{M_1}U, \quad v_B = \bar{v} + V, \quad w_B = W,$$

$$p_B = \bar{p} + \frac{1}{R}(\tilde{p} + P), \quad T_B = \bar{T} + T, \tag{1}$$

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where

$$\bar{u} = \frac{\partial f}{\partial \eta}, \quad \bar{v} = \frac{1}{\sqrt{2x}} \left(-\bar{T}f + \frac{\partial f}{\partial \eta} I(\bar{T}) \right) + \sqrt{2x} \frac{\partial I(\bar{T})}{\partial x},$$

$$f = \eta - \frac{\beta}{M^{1/2}} + \frac{\bar{f}(x,\eta)}{M_1} + \cdots,$$

$$\bar{T} = \bar{T}(x,\eta) + o(M^0), \tag{2}$$

$$\frac{\partial \bar{f}}{\partial \eta} = \bar{f}_0(x,\eta_2) + \frac{\partial \bar{f}_0}{\partial \eta}(x,\eta_2) \cdot \xi \epsilon^{2/3} + \epsilon^{4/3} \hat{f}_2(x,\xi) + \cdots,$$

$$\bar{T} = \bar{T}_0(x,\eta_2) + \frac{\partial \bar{T}_0}{\partial \eta}(x,\eta_2) \cdot \xi \epsilon^{2/3} + \epsilon^{4/3} \hat{T}_2(x,\xi) + \cdots.$$

Here $\bar{f}_0$ and $\hat{f}_2$ are respectively the same as those appearing in (18a) and (38f) and $I(\bar{T})$ is defined by (5). The harmonic part (U, V, W, P, T) expands as in (38). But without loss of generality we rewrite them as

$$U = \epsilon^{4/3} \cos \frac{z}{\epsilon} (U_{01} + \epsilon^{2/3} U_{11} + \cdots) + \cdots,$$

$$V = \epsilon^{-2/3} \cos \frac{z}{\epsilon} (V_{01} + \epsilon^{2/3} V_{11} + \cdots) + \cdots,$$

$$W = \epsilon^{-1/3} \sin \frac{z}{\epsilon} (W_{01} + \epsilon^{2/3} W_{11} + \cdots) + \cdots, \tag{3}$$

$$T = \epsilon^{4/3} \cos \frac{z}{\epsilon} (T_{01} + \epsilon^{2/3} T_{11} + \cdots) + \cdots,$$

$$P = \epsilon^{-4/3} \cos \frac{z}{\epsilon} (P_{01} + \epsilon^{2/3} P_{11} + \cdots) + \cdots,$$

Comparing (3) with (38) shows that U, V, W, T, P here are in turn equal to $2U, 2V, 2iW, 2T, 2P$ there. Therefore, the equation satisfied by V_{01} here can be obtained by replacing V_{01} in (48) by $V_{01}/2$. Thus we have

$$\frac{\partial^2 V_{01}}{\partial \xi^2} + S_1(x)\xi V_{01} = \frac{\sigma^2 x}{3\bar{\mu}_0^2} V_{01}^3 + S_2(x) V_{01}, \tag{4}$$

We now look for travelling wave solutions superimposed on the above steady state. The total flow is then written as

$$u = u_B + \frac{1}{M_1} \delta U^*, \quad v = v_B + \delta V^*, \quad w = w_B + \delta W^*,$$

$$p = p_B + \delta P^*, \quad T = T_B + \delta T^*, \tag{5}$$

where δ is a small parameter introduced to facilitate linearization. The linearized perturbation equations are obtained by replacing (U, V, W, P, T) in (11)–(15) by $(U + \delta U^*, V + \delta V^*, W + \delta W^*, P + \delta P^*, T + \delta T^*)$, putting back the $\partial/\partial t$ terms in the momentum and energy balance equations, and then linearizing in terms of δ. They are given by

$$\frac{1}{\bar{T}} \left(\frac{\partial U^*}{\partial x} + \frac{\bar{f}'}{M_1} \frac{\partial U^*}{\partial x} - \frac{\eta}{2x} \frac{\partial U^*}{\partial \eta} \right) - \bar{\mu} U_{zz}^* - \frac{1}{2x\bar{T}} \frac{\partial}{\partial \eta} \left(\frac{\mu}{\bar{T}} \frac{\partial U^*}{\partial \eta} \right) + \frac{\bar{f}''}{\sqrt{2x}\bar{T}^2} V^*$$

$$+[\frac{\eta\bar{f}''}{2x\bar{T}^2} - \frac{1}{2x\bar{T}}\frac{\partial}{\partial\eta}(\frac{\tilde{\mu}\bar{f}''}{\bar{T}})]T^* - \frac{\tilde{\mu}}{2x\bar{T}^2}\bar{f}''\frac{\partial T^*}{\partial\eta}$$

$$+\frac{1}{\bar{T}}[\frac{1}{\sqrt{2x\bar{T}}}(V^*\frac{\partial U}{\partial\eta} + V\frac{\partial U^*}{\partial\eta}) + W^*U_z + WU_z^*]$$

$$-\frac{T}{\bar{T}^2}[\frac{\partial U^*}{\partial x} - \frac{\eta}{2x}\frac{\partial U^*}{\partial\eta} + \frac{\bar{f}''}{\sqrt{2x\bar{T}}}V^*] - \frac{T^*}{\bar{T}^2}[\frac{\partial U}{\partial x} - \frac{\eta}{2x}\frac{\partial U}{\partial\eta} + \frac{\bar{f}''}{\sqrt{2x\bar{T}}}V]$$

$$-\frac{1}{2x\bar{T}}\frac{\partial}{\partial\eta}(\frac{\tilde{\mu}}{\bar{T}}T\frac{\partial U^*}{\partial\eta}) - \frac{1}{2x\bar{T}}\frac{\partial}{\partial\eta}(\frac{\tilde{\mu}}{\bar{T}}T^*\frac{\partial U}{\partial\eta}) - \tilde{\mu}(TU_z^*)_z - \tilde{\mu}(T^*U_z)_z$$

$$(\frac{\partial\bar{f}'}{\partial x} - \frac{\eta\bar{f}''}{2x})\frac{2TT^*}{\bar{T}^3} - \frac{1}{2x\bar{T}}\frac{\partial}{\partial\eta}\left(\frac{\tilde{\mu}\bar{f}''}{\bar{T}}TT^*\right) + (\frac{1}{\bar{T}} - \frac{T}{\bar{T}^2})\frac{\partial U^*}{\partial t} = 0, \qquad (6)$$

$$\frac{1}{\bar{T}}(\frac{\partial V^*}{\partial x} + \frac{\bar{f}'}{M_1}\frac{\partial V^*}{\partial x} - \frac{\eta}{2x}\frac{\partial V^*}{\partial\eta}) - \bar{\mu}V_{zz}^* - \frac{\eta\bar{T}'}{2x\bar{T}^2}V^* - \frac{4}{3\cdot(2x)\bar{T}}\frac{\partial}{\partial\eta}(\frac{\mu}{\bar{T}}\frac{\partial V^*}{\partial\eta})$$

$$+\frac{1}{\sqrt{2x\bar{T}}}\frac{\partial P^*}{\partial\eta} - \frac{H(x)}{\bar{T}^2}\frac{T^*}{\epsilon^4} - \frac{2\tilde{\mu}\bar{v}'}{3x\bar{T}^2}\frac{\partial T^*}{\partial\eta} + \frac{2\tilde{\mu}\bar{T}'}{3\sqrt{2x\bar{T}}}W_z^*$$

$$-\frac{\bar{\mu}}{3\sqrt{2x\bar{T}}}\frac{\partial}{\partial\eta}(W_z^*) + \frac{1}{\bar{T}}\left(\frac{1}{\sqrt{2x\bar{T}}}[V^*V_\eta + VV_\eta^*] + W^*V_z + WV_z^*\right)$$

$$-\frac{T}{\bar{T}^2}\left(\frac{\partial V^*}{\partial x} - \frac{\eta}{2x}\frac{\partial V^*}{\partial\eta} - \frac{\eta\bar{T}'}{2x\bar{T}}V^*\right) - \frac{T^*}{\bar{T}^2}\left(\frac{\partial V}{\partial x} - \frac{\eta}{2x}\frac{\partial V}{\partial\eta} - \frac{\eta\bar{T}'}{2x\bar{T}}V\right)$$

$$-\left\{\frac{4}{3\cdot(2x)\bar{T}^2}\frac{\partial\tilde{\mu}}{\partial\eta}\frac{\partial V}{\partial\eta} + \frac{4\bar{\mu}}{3\cdot(2x)\bar{T}}\frac{\partial}{\partial\eta}(\frac{1}{\bar{T}}\frac{\partial V}{\partial\eta}) + \frac{\bar{\mu}}{3\sqrt{2x\bar{T}}}\frac{\partial W_z}{\partial\eta} - \frac{2}{3\sqrt{2x\bar{T}}}\frac{\partial\tilde{\mu}}{\partial\eta}W_z + \tilde{\mu}V_{zz}\right\}T^*$$

$$-\left\{\frac{4}{3\cdot(2x)\bar{T}^2}\frac{\partial\tilde{\mu}}{\partial\eta}\frac{\partial V^*}{\partial\eta} + \frac{4\bar{\mu}}{3\cdot(2x)\bar{T}}\frac{\partial}{\partial\eta}(\frac{1}{\bar{T}}\frac{\partial V^*}{\partial\eta}) + \frac{\bar{\mu}}{3\sqrt{2x\bar{T}}}\frac{\partial W_z^*}{\partial\eta}\right.$$

$$\left. -\frac{2}{3\sqrt{2x\bar{T}}}\frac{\partial\tilde{\mu}}{\partial\eta}W_z^* + \tilde{\mu}V_{zz}^*\right\}T - \left(\frac{4\bar{\mu}}{3\sqrt{2x\bar{T}}}\frac{\partial V}{\partial\eta} - \frac{2}{3}\tilde{\mu}W_z\right)\frac{1}{\sqrt{2x\bar{T}}}\frac{\partial T^*}{\partial\eta}$$

$$-\left(\frac{4\bar{\mu}}{3\sqrt{2x\bar{T}}}\frac{\partial V^*}{\partial\eta} - \frac{2}{3}\tilde{\mu}W_z^*\right)\frac{1}{\sqrt{2x\bar{T}}}\frac{\partial T}{\partial\eta} - \left(\frac{1}{\sqrt{2x\bar{T}}}\frac{\partial W}{\partial\eta} + V_z\right)\tilde{\mu}T_z^*$$

$$+\frac{H(x)}{\bar{T}^3}\cdot\frac{2TT^*}{\epsilon^4} - \frac{1}{6x\bar{T}}\frac{\partial}{\partial\eta}\left(\frac{\tilde{\mu}\bar{v}'}{\bar{T}}TT^*\right) - \left(\frac{1}{\sqrt{2x\bar{T}}}\frac{\partial W^*}{\partial\eta} + V_z^*\right)\tilde{\mu}T_z$$

$$+(\frac{1}{\bar{T}} - \frac{T}{\bar{T}^2})\frac{\partial V^*}{\partial t} = 0, \qquad (7)$$

$$-\frac{1}{\bar{T}}(\frac{\partial W^*}{\partial x} + \frac{\bar{f}'}{M_1}\frac{\partial W^*}{\partial x} - \frac{\eta}{2x}\frac{\partial W^*}{\partial\eta}) + \frac{\tilde{\mu}\bar{T}'}{\sqrt{2x\bar{T}}}V_z^* + \frac{\bar{\mu}}{3\sqrt{2x\bar{T}}}\frac{\partial V_z^*}{\partial\eta} - P_z^* + \frac{2\bar{\mu}\eta\bar{T}'}{3\cdot(2x)\bar{T}}T_z^*$$

$$+\frac{4}{3}\bar{\mu}W_{zz}^* + \frac{1}{2x\bar{T}}\frac{\partial}{\partial\eta}(\frac{\bar{\mu}}{\bar{T}}\frac{\partial W^*}{\partial\eta}) - \frac{1}{\bar{T}}\left(\frac{1}{\sqrt{2x\bar{T}}}V^*\frac{\partial W}{\partial\eta} + \frac{1}{\sqrt{2x\bar{T}}}V\frac{\partial W^*}{\partial\eta} + W^*W_z\right.$$

$$\left. +WW_z^*\right) + \frac{T^*}{\bar{T}^2}(\frac{\partial W}{\partial x} - \frac{\eta}{2x}\frac{\partial W}{\partial\eta}) + \frac{T}{\bar{T}^2}(\frac{\partial W^*}{\partial x} - \frac{\eta}{2x}\frac{\partial W^*}{\partial\eta})$$

$$+\left\{\frac{4}{3}\bar{\mu}W_{zz} + \frac{\bar{\mu}}{3\sqrt{2x\bar{T}}}\frac{\partial V_z}{\partial\eta} + \frac{1}{\sqrt{2x\bar{T}}}\frac{\partial\tilde{\mu}}{\partial\eta}V_z + \frac{1}{2x\bar{T}^2}\frac{\partial\tilde{\mu}}{\partial\eta}\frac{\partial W}{\partial\eta} + \frac{\bar{\mu}}{2x\bar{T}}\frac{\partial}{\partial\eta}(\frac{1}{\bar{T}}\frac{\partial W}{\partial\eta})\right\}T^*$$

$$+\left\{\frac{4}{3}\bar{\mu}W_{zz}^* + \frac{\bar{\mu}}{3\sqrt{2x\bar{T}}}\frac{\partial V_z^*}{\partial\eta} + \frac{1}{\sqrt{2x\bar{T}}}\frac{\partial\tilde{\mu}}{\partial\eta}V_z^* + \frac{1}{2x\bar{T}^2}\frac{\partial\tilde{\mu}}{\partial\eta}\frac{\partial W^*}{\partial\eta} + \frac{\bar{\mu}}{2x\bar{T}}\frac{\partial}{\partial\eta}(\frac{1}{\bar{T}}\frac{\partial W^*}{\partial\eta})\right\}T$$

$$+\frac{\bar{\mu}}{\sqrt{2x\bar{T}}}(\frac{1}{\sqrt{2x\bar{T}}}\frac{\partial W}{\partial \eta}+V_z)\frac{\partial T^*}{\partial \eta}+\frac{\bar{\mu}}{\sqrt{2x\bar{T}}}(\frac{1}{\sqrt{2x\bar{T}}}\frac{\partial W^*}{\partial \eta}+V_z^*)\frac{\partial T}{\partial \eta}$$

$$+\left[\frac{4}{3}W_z-\frac{2}{3\sqrt{2x\bar{T}}}\frac{\partial V}{\partial \eta}\right]\bar{\mu}T_z^*+\left[\frac{4}{3}W_z^*-\frac{2}{3\sqrt{2x\bar{T}}}\frac{\partial V^*}{\partial \eta}\right]\bar{\mu}T_z-(\frac{1}{T}-\frac{T}{T^2})\frac{\partial W^*}{\partial t}=0, \quad (8)$$

$$-(\frac{\partial}{\partial x}+\frac{\bar{f}'}{M_1}\frac{\partial}{\partial x}-\frac{\eta}{2x}\frac{\partial}{\partial \eta})(\frac{T^*}{\bar{T}^2})+\frac{1}{\sqrt{2x\bar{T}}}\frac{\partial}{\partial \eta}(\frac{V^*}{\bar{T}})+(\frac{W^*}{\bar{T}})_z-\frac{\bar{v}'}{\sqrt{2x\bar{T}^3}}\cdot T^*$$

$$-(\frac{1}{\sqrt{2x\bar{T}}}\frac{\partial V}{\partial \eta}+W_z)\frac{T^*}{\bar{T}^2}-(\frac{1}{\sqrt{2x\bar{T}}}\frac{\partial V^*}{\partial \eta}+W_z^*)\frac{T}{\bar{T}^2}-\frac{\partial}{\partial t}(\frac{T^*}{\bar{T}^2})$$

$$-(\frac{1}{\sqrt{2x\bar{T}}}V\frac{\partial}{\partial \eta}+W\frac{\partial}{\partial z})(\frac{T^*}{\bar{T}^2})-(\frac{1}{\sqrt{2x\bar{T}}}V^*\frac{\partial}{\partial \eta}+W^*\frac{\partial}{\partial z})(\frac{T}{\bar{T}^2})$$

$$+(\frac{\partial}{\partial x}-\frac{\eta}{2x}\frac{\partial}{\partial \eta})(\frac{2TT^*}{\bar{T}^3})+\frac{2\bar{v}'}{\sqrt{2x\bar{T}^4}}TT^*=0, \quad (9)$$

$$\frac{1}{\bar{T}}(\frac{\partial T^*}{\partial x}+\frac{\bar{f}'}{M_1}\frac{\partial T^*}{\partial x}-\frac{\eta}{2x}\frac{\partial T^*}{\partial \eta})+\frac{\bar{T}'}{\sqrt{2x\bar{T}^2}}V^*+\left\{\frac{\eta\bar{T}'}{2x\bar{T}^2}-\frac{1}{2x\sigma\bar{T}}\frac{\partial}{\partial \eta}(\frac{\bar{\mu}\bar{T}'}{\bar{T}})\right\}T^*$$

$$-\frac{\bar{\mu}}{\sigma}T_{zz}^*-\frac{\bar{\mu}\bar{T}'}{2x\bar{T}^2\sigma}\frac{\partial T^*}{\partial \eta}-\frac{1}{2x\bar{T}\sigma}\frac{\partial}{\partial \eta}(\frac{\bar{\mu}}{\bar{T}}\frac{\partial T^*}{\partial \eta})$$

$$+\frac{1}{\bar{T}}\left(\frac{1}{\sqrt{2x\bar{T}}}V^*\frac{\partial T}{\partial \eta}+\frac{1}{\sqrt{2x\bar{T}}}V\frac{\partial T^*}{\partial \eta}+W^*T_z+WT_z^*\right)$$

$$-\frac{T}{\bar{T}^2}\left(\frac{\partial T^*}{\partial x}-\frac{\eta}{2x}\frac{\partial T^*}{\partial \eta}+\frac{\bar{T}'}{\sqrt{2x\bar{T}}}V^*\right)-\frac{T^*}{\bar{T}^2}\left(\frac{\partial T}{\partial x}-\frac{\eta}{2x}\frac{\partial T}{\partial \eta}+\frac{\bar{T}'}{\sqrt{2x\bar{T}}}V\right)$$

$$-\frac{1}{2x\bar{T}^2\sigma}\frac{\partial\bar{\mu}}{\partial \eta}T\frac{\partial T^*}{\partial \eta}-\frac{1}{2x\bar{T}^2\sigma}\frac{\partial\bar{\mu}}{\partial \eta}T^*\frac{\partial T}{\partial \eta}-\frac{\bar{\mu}}{\sigma}\{(T^*T_z)_z+(TT_z^*)_z\}$$

$$-\frac{\bar{\mu}}{2x\bar{T}\sigma}\frac{\partial}{\partial \eta}(\frac{T}{\bar{T}}\cdot\frac{\partial T^*}{\partial \eta}+\frac{T^*}{\bar{T}}\cdot\frac{\partial T}{\partial \eta})+(\frac{\partial\bar{T}}{\partial x}-\frac{\eta\bar{T}'}{2x})\frac{2TT^*}{\bar{T}^3}$$

$$-\frac{1}{2x\sigma\bar{T}}\frac{\partial}{\partial \eta}\left(\frac{\tilde{\mu}\bar{T}'}{\bar{T}}TT^*\right)+(\frac{1}{T}-\frac{T}{\bar{T}^2})\frac{\partial T^*}{\partial t}=0. \quad (10)$$

Here $\bar{f}'$ has been used to denote $\partial\bar{f}/\partial\eta$ to simplify notation. In the two transition layers, the vertical lengthscale is characterized by the variable ξ defined by

$$\xi=\frac{\eta-\eta_2}{\epsilon^{2/3}}, \quad (11)$$

thus

$$\frac{\partial}{\partial \eta}=O(\epsilon^{-2/3}), \quad \frac{\partial}{\partial x}=O(\epsilon^{-2}), \quad \frac{\partial}{\partial t}=O(\epsilon^{-2}), \quad (12)$$

where the scales for x and t are deduced from perturbatin equations. Since we are looking for travelling wave solutions, it is convenient to define a new variable ϕ by

$$\phi=t-x \quad (13)$$

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and use (t, x, ξ, z) in place of (ϕ, x, ξ, z) as the new independent variables. Hence

$$\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x} - \frac{\partial}{\partial \phi}, \quad \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial \phi},$$

and the operator

$$\left(1 + \frac{\bar{f}'}{M_1}\right) \frac{\partial}{\partial x} + \frac{\partial}{\partial t}$$

which appears in (6)–(10) is transformed to

$$\frac{\partial}{\partial x} - \frac{\bar{f}'}{M_1} \frac{\partial}{\partial \phi} \tag{14}$$

to leading order.

Next, we assume that the superimposed disturbance is $\pi/2$ out of phase with the steady state. Therefore, the perturbation quantities take the form

$$U^* = \epsilon^{4/3} \sin\frac{z}{\epsilon} \cdot E \cdot (u_{01} + \epsilon^{2/3} u_{11} + \cdots) + \cdots + C.C.,$$

$$V^* = \epsilon^{-2/3} \sin\frac{z}{\epsilon} \cdot E \cdot (v_{01} + \epsilon^{2/3} v_{11} + \cdots) + \cdots + C.C.,$$

$$W^* = \epsilon^{-1/3} E \cdot (w_{m0} + \epsilon^{2/3} w_{m1} + \cdots) + \cdots$$

$$+ \epsilon^{-1/3} \cos\frac{z}{\epsilon} \cdot E \cdot (w_{01} + \epsilon^{2/3} w_{11} + \cdots) + \cdots + C.C., \tag{15}$$

$$P^* = \epsilon^{-4/3} \sin\frac{z}{\epsilon} \cdot E \cdot (p_{01}^* + \epsilon^{2/3} p_{11}^* + \cdots) + \cdots + C.C.,$$

$$T^* = \epsilon^{4/3} \sin\frac{z}{\epsilon} \cdot E \cdot (\theta_{01}^* + \epsilon^{2/3} \theta_{11}^* + \cdots) + \cdots + C.C..$$

Here

$$E = \exp\left(\frac{i}{\epsilon^2} \int^x K(x) dx - \frac{iM_1 \Omega \phi}{\epsilon^2}\right), \tag{16}$$

where Ω is the constant frequency and the wavenumber K expands as

$$K = K_0 + \epsilon^{2/3} K_1 + \cdots. \tag{17}$$

The scale for ϕ in (16) is chosen so that in (14)

$$\frac{1}{M_1} \frac{\partial}{\partial \phi} = O(\frac{\partial}{\partial x}).$$

We now substitute (15) and (3) into (6)–(10). By equating the coefficients of $\epsilon^{-8/3} E \sin(z/\epsilon)$ and $\epsilon^{-2} E \sin(z/\epsilon)$ in (6) we obtain

$$\frac{i}{T_0}(K_0 + \bar{f}_0 \Omega) v_{01} + \bar{\mu}_0 v_{01} - \frac{H(x)}{T_0^2} \theta_{01}^* = 0, \tag{18}$$

$$\frac{i}{T_0}(K_0 + \bar{f}_0 \Omega) v_{11} + \frac{i}{T_0}[K_1 + \bar{f}_0' \xi \Omega - (K_0 + \bar{f}_0 \Omega)\frac{T_0'}{T_0}\xi] v_{01}$$

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$$+\bar{\mu}_0 v_{11} + \bar{T}'_0 \xi \bar{\mu}_0 v_{01} - \frac{4\bar{\mu}_0}{3 \cdot (2x)\bar{T}_0^2}\frac{\partial^2 v_{01}}{\partial\xi^2} + \frac{1}{\sqrt{2x}\bar{T}_0}\frac{\partial p^*_{01}}{\partial\xi}$$

$$- \frac{H(x)}{\bar{T}_0^2}(\theta^*_{11} - \frac{2\bar{T}'_0}{\bar{T}_0}\xi\theta^*_{01}) + \frac{\bar{\mu}_0}{3\sqrt{2x}\bar{T}_0}\frac{\partial w_{01}}{\partial\xi} - \frac{1}{\bar{T}_0}w_{m0}V_{01} = 0. \tag{19}$$

Equating the coefficients of $\epsilon^{-7/3}E\cos(z/\epsilon)$, $E\epsilon^{-7/3}$, and $E\epsilon^{-5/3}$ in (8) gives

$$- \frac{i}{\bar{T}_0}(K_0 + \bar{f}_0\Omega)w_{01} + \frac{\bar{\mu}_0}{3\sqrt{2x}\bar{T}_0}\frac{\partial v_{01}}{\partial\xi} - p^*_{01} - \frac{4}{3}\bar{\mu}_0 w_{01} = 0, \tag{20}$$

$$- \frac{i}{\bar{T}_0}(K_0 + \bar{f}_0\Omega)w_{m0} = 0, \tag{21}$$

$$- \frac{i}{\bar{T}_0}(K_0 + \bar{f}_0\Omega)w_{m1} - \frac{i}{\bar{T}_0}\left\{K_1 + \bar{f}'_0\xi\Omega - (K_0 + \bar{f}_0\Omega)\frac{\bar{T}'_0}{\bar{T}_0}\xi\right\}w_{m0}$$

$$+ \frac{\bar{\mu}_0}{2x\bar{T}_0^2}\frac{\partial^2 w_{m0}}{\partial\xi^2} - \frac{1}{2\sqrt{2x}\bar{T}_0^2}(v_{01}\frac{\partial W_{01}}{\partial\xi} + V_{01}\frac{\partial w_{01}}{\partial\xi}) = 0. \tag{22}$$

Finally, from equatig the coefficients of $\epsilon^{-4/3}E\sin(z/\epsilon)$ in (9), $\epsilon^{-2/3}E\sin(z/\epsilon)$ and $\epsilon^0 E\sin(z/\epsilon)$ in (10), we have

$$w_{01} - \frac{1}{\sqrt{2x}\bar{T}_0}\frac{\partial v_{01}}{\partial\xi} = 0, \tag{23}$$

$$\frac{i}{\bar{T}_0}(K_0 + \bar{f}_0\Omega)\theta^*_{01} + \frac{\bar{T}'_0}{\sqrt{2x}\bar{T}_0^2}v_{01} + \frac{1}{\sigma}\bar{\mu}_0\theta^*_{01} = 0, \tag{24}$$

$$\frac{i}{\bar{T}_0}(K_0 + \bar{f}_0\Omega)\theta^*_{11} + \frac{i}{\bar{T}_0}[K_1 + \bar{f}'_0\xi\Omega - (K_0 + \bar{f}_0\Omega)\frac{\bar{T}'_0}{\bar{T}_0}\xi]\theta^*_{01}$$

$$+ \frac{1}{\sqrt{2x}\bar{T}_0^2}\left[\bar{T}'_0 v_{11} + (\frac{\partial\hat{T}_2}{\partial\xi} - \frac{2\bar{T}'^2_0}{\bar{T}_0}\xi)v_{01}\right] - \frac{\bar{\mu}_0}{2x\sigma\bar{T}_0^2}\frac{\partial^2\theta^*_{01}}{\partial\xi^2}$$

$$+ \frac{1}{\sigma}(\bar{\mu}_0\theta^*_{11} + \bar{\mu}_0\bar{T}'_0\xi\theta^*_{01}) - \frac{1}{\bar{T}_0}w_{m0}\theta_{01} = 0. \tag{25}$$

We now proceed to solve this hierarchy of equations. First, it can be seen that (18) and (24) have non-trivial solutions only if

$$K_0 = -\bar{f}_0\Omega. \tag{26}$$

It then follows from (16) and (2a) that the travelling wave propagates downstream with the same speed to leading order as the that of the basic steady flow. We also note that (21) is now automatically satisfied.

From (24), (23) and (20) we have

$$\theta^*_{01} = -\frac{\sigma\bar{T}'_0}{\sqrt{2x}\bar{\mu}_0\bar{T}_0^2}v_{01}, \quad w_{01} = \frac{1}{\sqrt{2x}\bar{T}_0}\frac{\partial v_{01}}{\partial\xi}, \quad p^*_{01} = -\frac{\bar{\mu}_0}{\sqrt{2x}\bar{T}_0}\frac{\partial v_{01}}{\partial\xi}. \tag{27}$$

With the use of these relations, equation (19) and (25) can be reduced to a 2 × 2 matrix equation of the form $\mathbf{A}\zeta = \mathbf{f}$ for $\zeta = (v_{11}, \theta^*_{11})^T$. The inhomogeneous term $\mathbf{f}$ involves v_{01} and its derivatives. It also involves $\partial\hat{T}_2/\partial\xi$, the expression of which can be obtained from (47) by

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replacing V_{01} there by $V_{01}/2$. Since the coefficient matrix $\mathbf{A}$ has zero determinant, ζ has a non-trivial solution only if the inner product of $\mathbf{f}$ with the left eigenvector of $\mathbf{A}$ vanishes. The condition then leads, after some algebra, to the equation

$$\frac{\partial^2 v_{01}}{\partial \xi^2} - \frac{(2x)\bar{T}_0^2}{3\bar{\mu}_0}\left\{\sigma\eta_2\bar{T}_0\xi + 3(\tilde{\mu}_0 + \frac{\bar{\mu}_0}{\bar{T}_0})\bar{T}_0'\xi + \frac{i(1+\sigma)}{\bar{T}_0}(K_1 + \bar{f}_0'\xi\Omega)\right.$$

$$\left. + \frac{\sigma^2}{2\bar{\mu}_0\bar{T}_0^2}V_{01}^2 + \frac{\bar{\mu}_0}{\bar{T}_0'}S(x)\right\}v_{01} + \frac{(1+\sigma)(2x)\bar{T}_0}{3\bar{\mu}_0}w_{m0}V_{01} = 0. \tag{28}$$

The equation involves w_{m0} as well as V_{01}. To determine w_{m0}, we turn to (22) which can be shown to reduce to

$$\frac{\partial^2 w_{m0}}{\partial \xi^2} - i\cdot\frac{2x\bar{T}_0}{\bar{\mu}_0}(K_1 + \bar{f}_0'\xi\Omega)w_{m0} + \frac{1}{2\bar{\mu}_0\bar{T}_0}\left(v_{01}\frac{\partial^2 V_{01}}{\partial\xi^2} - V_{01}\frac{\partial^2 v_{01}}{\partial\xi^2}\right) = 0. \tag{29}$$

With the use of the definitions (49) and (50), equation (28) can be rewriten as

$$\frac{\partial^2 v_{01}}{\partial\xi^2} + S_1(x)\xi v_{01} = \frac{(2x)\sigma^2}{6\bar{\mu}_0^2}V_{01}^2 v_{01} + S_2(x)v_{01}$$

$$+ \frac{(1+\sigma)(2x)\bar{T}_0}{3\bar{\mu}_0}\left\{(iK_1 + i\Omega\bar{f}_0'\xi)v_{01} - w_{m0}V_{01}\right\}. \tag{30}$$

Finally, after (4) and (30) have been used to eliminate the second order derivatives of V_{01} and v_{01}, equation (29) becomes

$$\frac{\partial^2 w_{m0}}{\partial\xi^2} - \frac{2x\bar{T}_0}{\bar{\mu}_0}(iK_1 + i\Omega\bar{f}_0'\xi)w_{m0} = \frac{(1+\sigma)(2x)}{6\bar{\mu}_0^2}\left\{(iK_1 + i\Omega\bar{f}_0'\xi)V_{01}v_{01} - w_{m0}V_{01}^2\right\}. \tag{31}$$

Equations (30) and (31) are to be solved simultaneously to determine the second order correction K_1 to the wavenumber and the frequency Ω, subject to the conditions that v_{01}, $w_{m0} \to 0$ as $\eta \to \pm\infty$ so that the travelling waves are confined within the transition layers. It is possible to simplify these two equations by scaling the flow properties of the basic steady state out of this eigenvalue problem. This can be achieved by introducing the following new independent and dependent variables:

$$\zeta = -S_1^{\frac{1}{3}}(x)(\xi - \frac{S_2(x)}{S_1(x)}), \quad V = -\frac{\sigma\sqrt{x}}{\sqrt{3}\bar{\mu}_0 S_1^{1/3}(x)}\cdot V_{01},$$

$$v = -\frac{\sigma}{1+\sigma}\sqrt{\frac{2}{x}}\cdot\frac{S_1^{1/3}}{\bar{T}_0}\cdot v_{01}, \quad w = w_{m0}. \tag{32}$$

In terms of these new variables, equations (4), (30) and (31) become

$$\frac{d^2 V}{d\zeta^2} - \zeta V = V^3, \tag{33}$$

$$\frac{d^2 v}{d\zeta^2} - (1 + \frac{2}{3}i\hat{\Omega})\zeta v - \frac{2}{3}i\hat{K}v - vV^2 + \frac{2}{3}\sqrt{6}Vw = 0, \tag{34}$$

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$$\frac{d^2w}{d\zeta^2} - \frac{2}{1+\sigma} \cdot i(\hat{\Omega}\zeta + \hat{K})w + \frac{1+\sigma}{\sigma^2}V^2w - \frac{1+\sigma}{\sqrt{6}\sigma^2} \cdot i(\Omega\zeta + \hat{K})Vv = 0, \tag{35}$$

where

$$\hat{\Omega} = -\frac{(1+\sigma)\bar{T}_0\bar{f}_0'x}{\bar{\mu}_0 S_1(x)} \cdot \Omega, \quad \hat{K} = \frac{(1+\sigma)\bar{T}_0 x}{\bar{\mu}_0 S_1^{2/3}(x)} \cdot \left(K_1 + \frac{\bar{f}_0' S_2(x)}{S_1(x)} \cdot \Omega \right). \tag{36}$$

Equations (33)–(35) are of the same form as their counterparts for incompressible flows as discussed by Hall and Seddougui (1989) and Seddougui and Bassom (1990). In fact, when $\sigma = 1$, the two eigenvalue problems are identical. The neutrally stable solutions correspond to real values of $\hat{\Omega}$ and $\hat{K}$. Such solutions were first given by Hall and Seddougui (1989) and were later improved upon by Bassom and Seddougui (1990). The latter authours' numerical solution shows that the lowest neutrally stable wavy mode has its eigenvalue pair given by

$$(\hat{K}, \hat{\Omega}) = (0.690, 0.372). \tag{37}$$

Since $\hat{\Omega} = 0$ corresponds to a stable wavy mode, when $\hat{\Omega}$ is increased the mode described by (37) is more dangerous than any other higher mode because it will occur first. It was conjectured by the above authors that in general there will be an infinite number of such neutral solutions. This conjecture was further supported by Bassom and Seddougui' (1990) asymptotic analysis which shows that there is indeed a family of neutral modes for $\hat{K} \gg 1$, $\hat{\Omega} \gg 1$.

Once the numerical values of $\hat{K}$ and $\hat{\Omega}$ have been found, the second order correction K_1 to the wavenumber and the frequency Ω can be determined from (36). Note that K_1 is a function of x. Thus for a given frequency Ω, $K_1(x)$ is the wavenumber for the travelling wave to be neutrally stable at x. If Ω is held fixed to be the neutral value at $x = \bar{x}$, then $K_1(x)$ will be complex when $x \neq \bar{x}$, implying that the travelling wave will experience spatial amplification or decay away from the neutral position.

The numercal values of K_1 and Ω are also dependent on the properties of the underlying steady state, the solution of which has been shown in sections 4 and 5. We shall not give any definite values for K_1 and Ω for any specific conditions, since the principal aim of the present section is to show that neutrally stable travelling wave solutions do exist in the present hypersonic context. If necessity comes, the values of K_1 and Ω for any specific situation are obtainable by using the relevant formulae given in the present paper.

7 Conclusion

In this paper, we have given an asymptotic description of the nonlinear development of large amplitude Görtler vortices downstream of the neutral positon. We have shown how an asymptotic state can be established under the combined effects of viscosity and nonlinearity. We have also investigated the possibility of such a large amplitude vortex structure losing stability to

travelling waves of the wavy type. Such an analysis has important applications, for example, to the flow in engine inlets and near the control surface of hypersonic vehicles.

The basis of our present studies is the linear theory given in our previous paper Fu, Hall and Blackaby (1990). It has been shown there that taking the large Mach number limit has two implications. Firstly, the boundary layer splits into two sublayers: a wall layer and a temperature adjustment layer. It is the latter layer that is most susceptible to Görtler vortices. We note that in their studies on the Rayleigh instability, Hall and Cowley (1990), Smith and Brown (1990) and Blackaby, Cowley and Hall (1990) found that the temperature adjustment layer is also most susceptible to the inviscid instability. Secondly, the boundary layer growth has two scales: a short scale related to the similarity variable η and the usual scale based on the streamwise variable x. The short scale is felt mainly through the $O(M^{3/2}/(2x)^{3/2})$ curvature of the basic state. Thus in the special case when the wall curvature is proportional to $(2x)^{-3/2}$, it exactly counterbalances the basic state curvature and Görtler vortices evolve downstream in the same manner as those in incompressible flows. In the more general curvature case, boundary layer growth strongly affects the evolution of Görtler vortices and it becomes negligible only when the local wavenumber is of $O(M^{3/8})$ or larger.

In the present paper, we have confined our attention to the $O(M^{3/8})$ wavenumber regime and thus we have been able to exclude the effects of boundary growth. The neutrally stable position is then uniquely defined. The linear theory tells us that when a certain parameter is positive, Görtler vortices will grow as they evolve downstream of the neutral position x_n. In the weakly nonlinear theory presented in section 3, we have determined the evolutionary behaviour of growing Görtler vortices in a small neighbourhood of the neutral position where Görtler vortices grow at a scale dictated by the variable $X = (x - x_n)/\epsilon$. It is shown that the mean temperature θ_{m0} and the first fundamental V_0 satisfy two coupled evolution equations; whilst the mean streamwise velocity u_{m0} can be determined from another evolution equation once θ_{m0} and V_0 have been found (other first fundamental components are related to V_0). In the limit $X \to \infty$, we have $\theta_{m0} \sim X^{3/2}$, $u_{m0} \sim X^{3/2}$ and $V_0 \sim X^{1/2}$ so that when $X = O(\epsilon^{-1})$, that is when $x - x_n = O(1)$, the mean temperature and streamwise velocity corrections become as large as the basic state. When this happens, the weakly nonlinear theory becomes invalid and the further downstream development of Görtler vortices is described by the fully nonlinear theory given in section 4. There it is shown that Görtler vortices spread into a region of $O(1)$ depth which is bounded by two transition layers. In the region of vortex activity, the mean temperature is determined from a solvability condition for the first fundamentals and thus it adjusts itself so as to make any modes neutrally stable everywhere simultaneously. The fact that the basic state is now completely altered by the the large amplitude Görtler vortices can be seen from (32) and (33) which show the first fundamental V_0^1 as a forcing to the "modified" basic state equations. In the two transition layers viscous effects make the fundamentals decay

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to zero exponentially, so that above the upper transition layer and below the lower transition layer there is only the mean flow. The centres of the two transition layers are determined by a free boundary problem, which has been solved numerically in section 5 for a number of curvature cases. Thus solutions for the first fundamentals and the mean flow quantities have been determined for $0 < \eta < \infty$ in closed form.

Once the large amplitude vortex structure described by the fully nonlinear theory has been established, transition can be reached by two possible routes in the form of secondary instabilities, as was shown by Swearingen and Blackwelder (1987). The first secondary instability is described here in section 6 which takes the form of time dependent travelling waves confined to the two transition layers and which leads to the wavy vortex boundaries observed experimentally. It is shown that such wavy type secondary instabilities may indeed exist in the present hypersonic context. The second possible secondary instability is associated with a Rayleigh instability. Relevant results will be given in our next paper.

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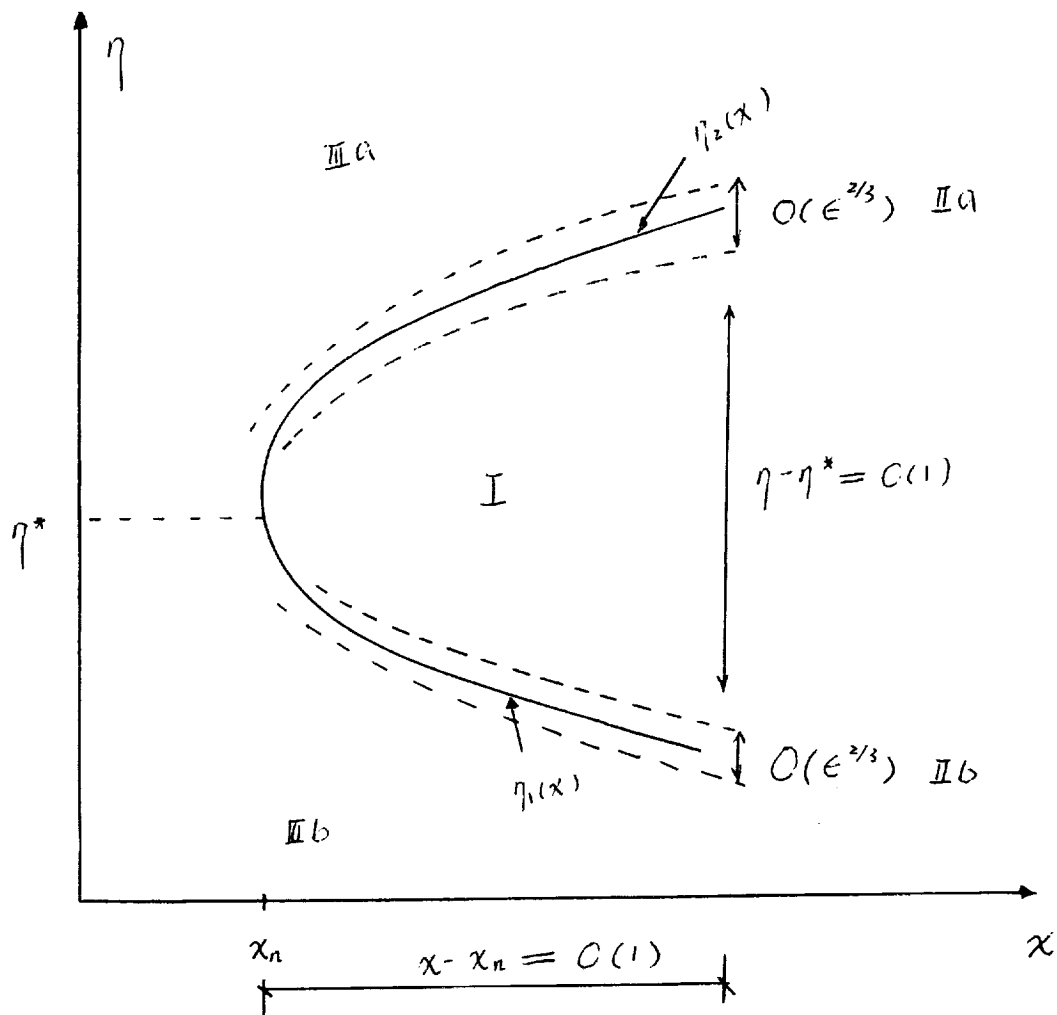
Fig.1

Figure 1. The different flow regions beyond the neutral stability position x_n.
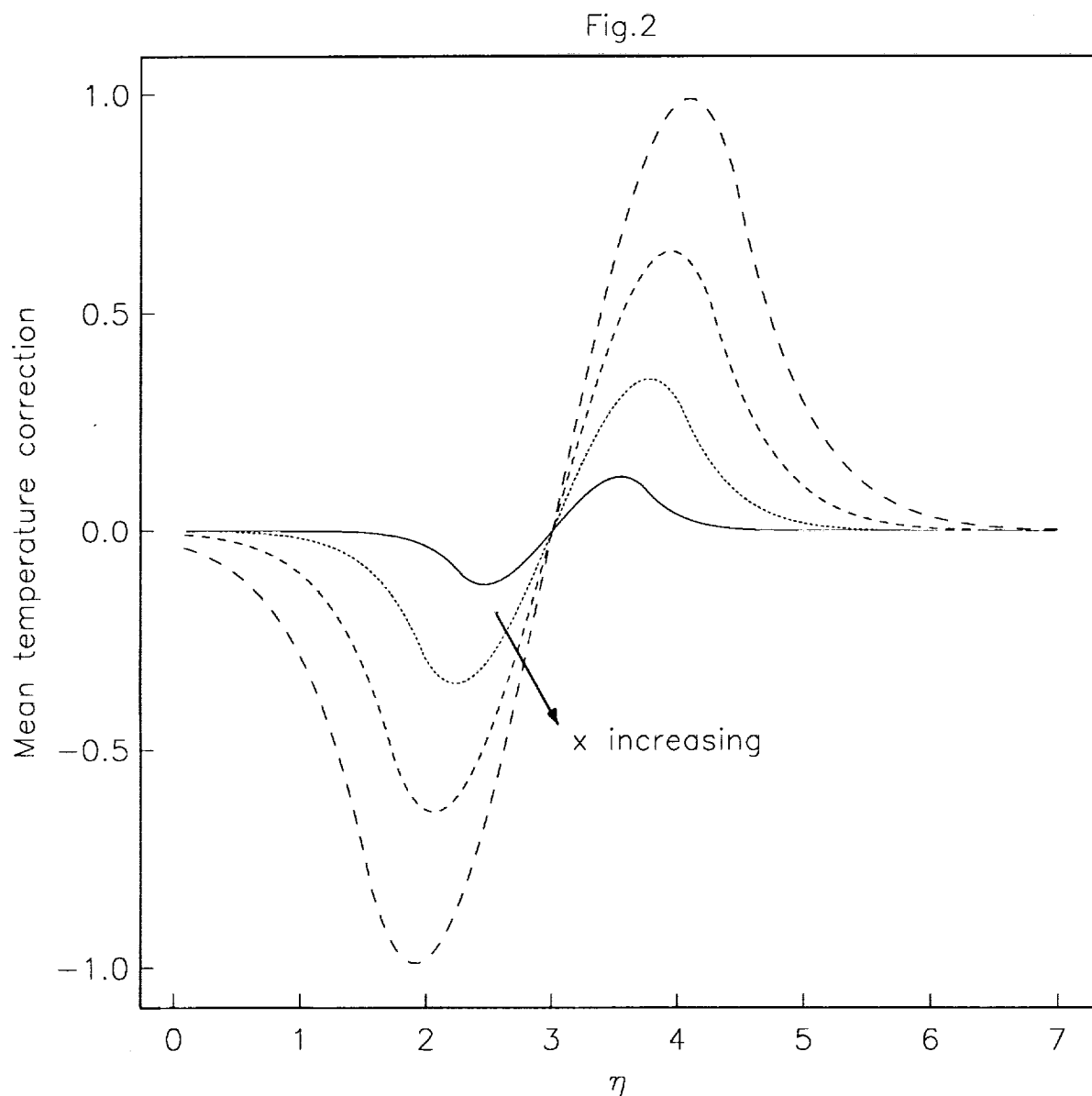
Fig.2

Figure 2. The growth of the mean temperature correction downstream of the neutral stability position $x_n = 0.4$ over a wall with curvature $\kappa = (2x)^{3/2}$, as predicted by the weakly nonlinear theory (5.56). The profiles shown correpond to x=0.6, 0.8, 1.0 and 1.2.
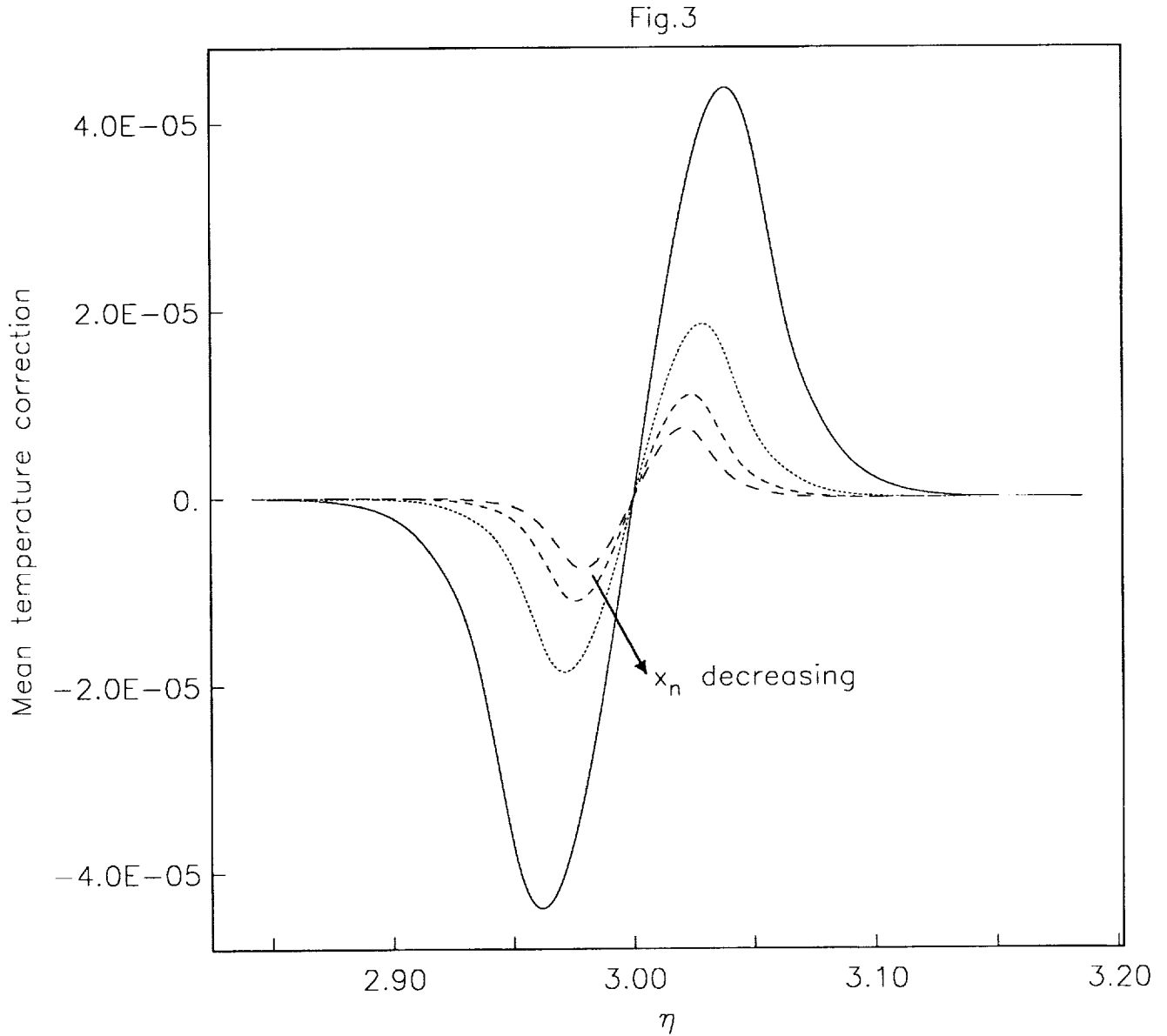
Fig.3

Figure 3. The dependence on x_n of the amplitude of the mean temperature correction at a fixed distance of 0.001 downstream of the neutral stability position x_n. The profiles shown correspond to $x_n = 0.4, 0.6, 0.8, 1.0$ and to the same curvature distribution as in Figure 2.
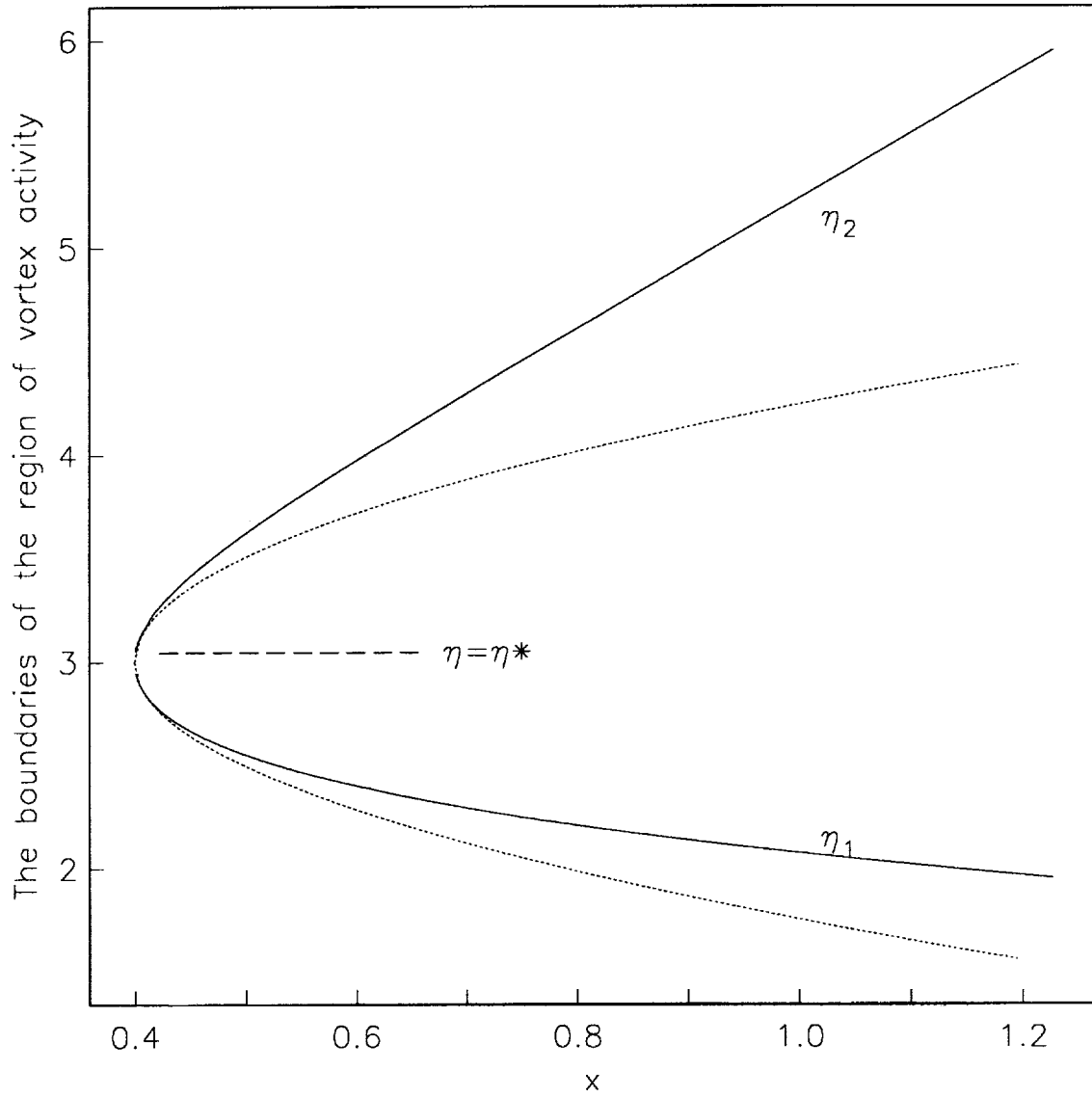
Fig.4

Figure 4. The development of η_1 and η_2 with x for the case $\kappa(x) = (2x)^{3/2}, x_n = 0.4$.
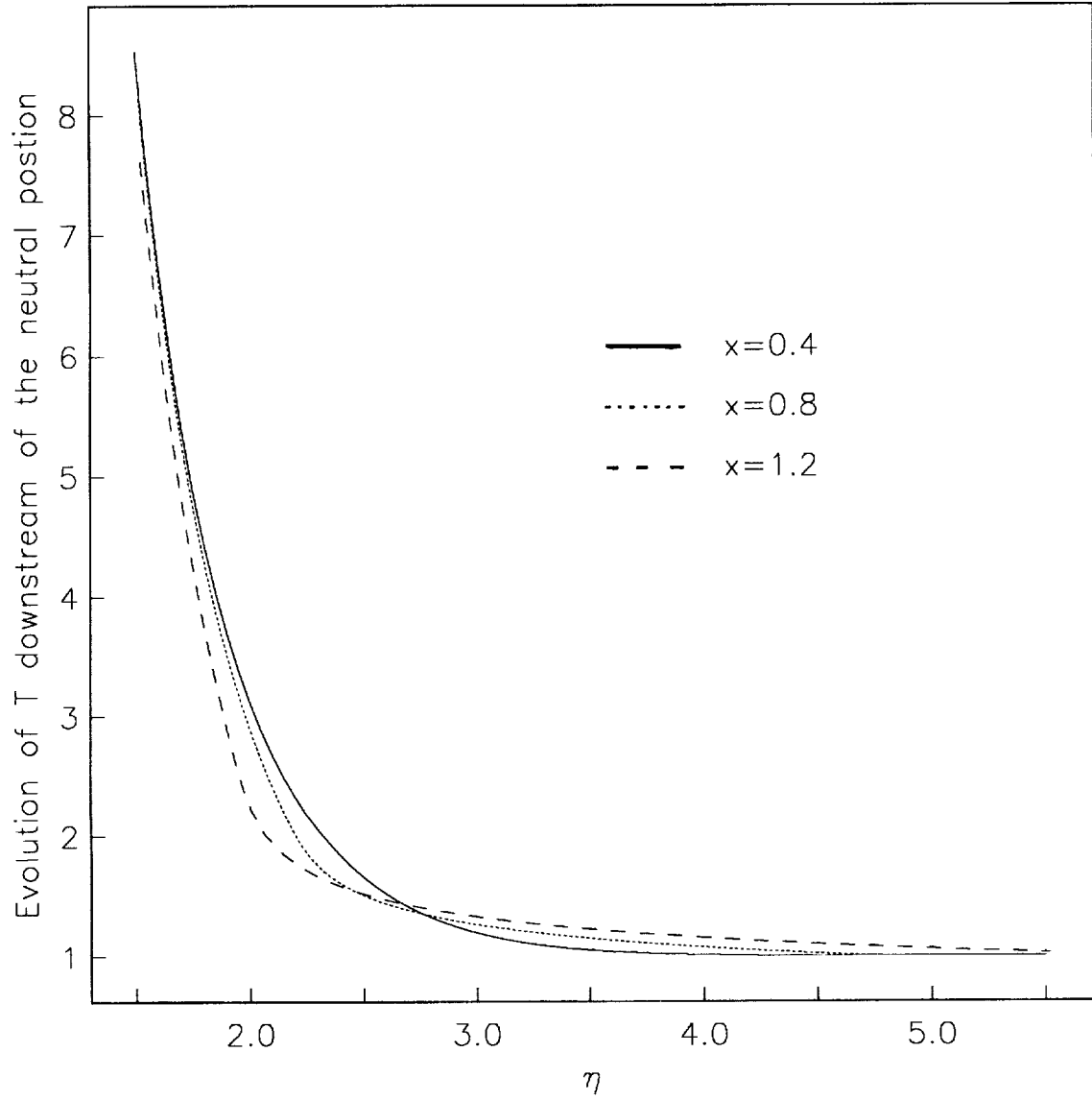———— fully nonlinear theory; $\cdots\cdots$ weakly nonlinear theory.

Fig.5

Figure 5 The non-harmonic part of the temperature at different downstream locations for the same case as in Figure 4.
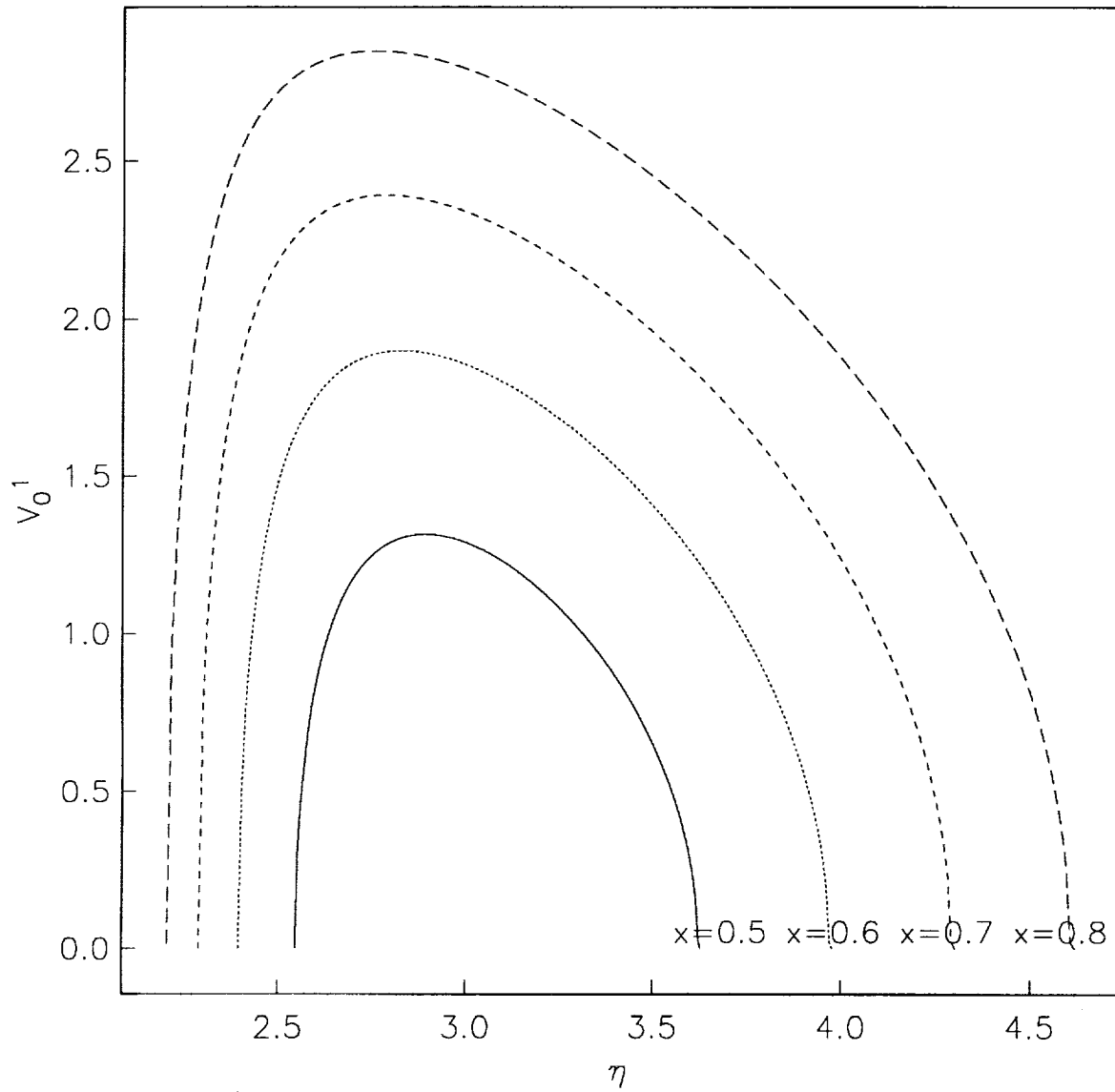
Fig.6

Figure 6 The eigenfunctions V_0^1 at different downstream locations for the same case as in Figure 4.
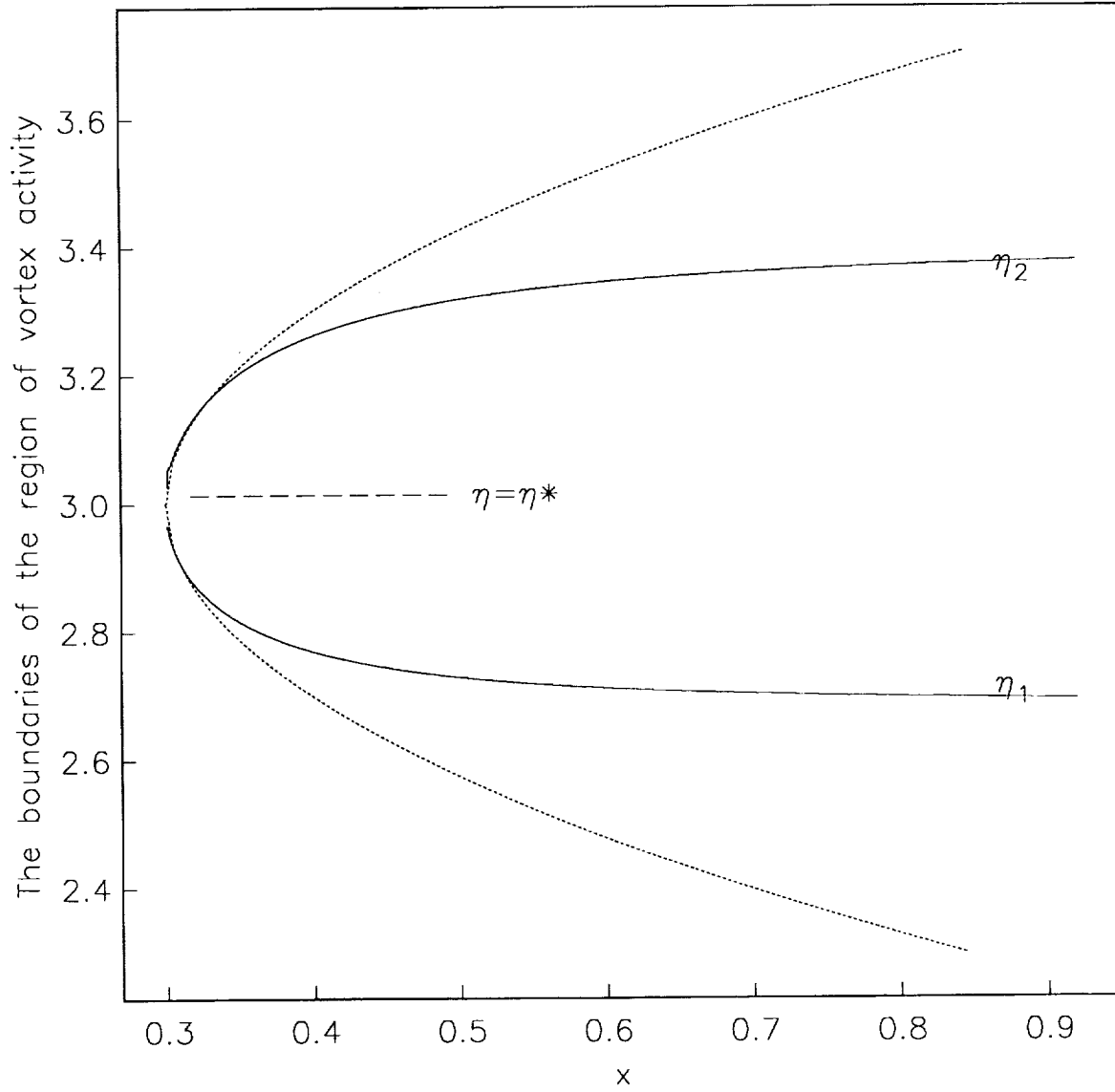
Fig.7

Figure 7. The development of η_1 and η_2 with x for the case $\kappa(x) = \sqrt{2x}, x_n = 0.3$.
——— fully nonlinear theory; $\cdots\cdots$ weakly nonlinear theory.

Report Documentation Page

1. Report No. NASA CR-187563 ICASE Report No. 91-39	2. Government Accession No.	3. Recipient's Catalog No.

4. Title and Subtitle NONLINEAR DEVELOPMENT AND SECONDARY INSTABILITY OF GÖRTLER VORTICES IN HYPERSONIC FLOWS	5. Report Date May 1991
	6. Performing Organization Code

7. Author(s) Yibin B. Fu Philip Hall	8. Performing Organization Report No. 91-39
	10. Work Unit No. 505-90-52-01

9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225	11. Contract or Grant No. NAS1-18605
	13. Type of Report and Period Covered Contractor Report

12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Langley Research Center Hampton, VA 23665-5225	14. Sponsoring Agency Code

15. Supplementary Notes
Langley Technical Monitor: Submitted to Euro. J. of Mechanics. Michael F. Card Final Report

16. Abstract

In a hypersonic boundary layer over a wall of variable curvature, the region most susceptible to Görtler vortices is the temperature adjustment layer over which the basic state temperature decreases monotonically to its free stream value (Hall & Fu (1989), Fu, Hall & Blackaby (1990)). Except for a special wall curvature distribution, the evolution of Görtler vortices trapped in the temperature adjustment layer will in general be strongly affected by boundary layer growth through the $O(M^{3/2})$ curvature of the basic state, where M is the free stream Mach number. Only when the local wavenumber becomes as large as of order $M^{3/8}$, do nonparallel effects become negligible in the determination of stability properties. In the latter case, Görtler vortices will be trapped in a thin layer of $O(\epsilon^{1/2})$ thickness which is embedded in the temperature adjustment layer; here ϵ is the inverse of the local wavenumber. In this paper, we first present a weakly nonlinear theory in which the initial nonlinear development of Görtler vortices in the neighbourhood of the neutral position is investigated and two coupled evolution equations are derived. From these two evolution equations we can determine whether the vortices are decaying or growing depending on the sign of a constant which is related to the wall curvature and the basic state temperature. In the latter case, it is found that the mean flow correction becomes as large as the basic state at distances $O(1)$ downstream of the neutral position. Next, we present a fully nonlinear theory concerning the further downstream development of these large-amplitude Görtler vortices. It is shown that the vortices spread out across the boundary layer. The upper and lower boundaries of the region of vortex activity are determined by a free-boundary problem involving the boundary layer equations. Finally, the secondary instability of the flow in the transition layers located at the upper and lower edges of the the region of vortex activity is considered. The superimposed wavy vortex perturbations are spanwise periodic travelling waves which are $\pi/2$ radians out of phase with the fundamental. The dispersion relation is found to be determined by solving two coupled differential equations and it is shown that an infinite number of neutrally stable modes may exist.

17. Key Words (Suggested by Author(s)) hypersonic, secondary instability	18. Distribution Statement 02 - Aerodynamics Unclassified - Unlimited

19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of pages 47	22. Price A03